A FAST FULL-WAVE SOLVER FOR THE ANALYSIS OF
LARGE PLANAR FINITE PERIODIC ANTENNA ARRAYS
IN GROUNDED MULTILAYERED MEDIA

DISSERTATION

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By

Pongsak Mahachoklertwattana, B.Eng, M.S.

* * * * *

The Ohio State University

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Dissertation Committee:

Prof. Prabhakar H. Pathak, Adviser
Prof. Jin-Fa Lee
Prof. Fernando L. Teixeira

Approved by

Adviser
Graduate Program in
Electrical and Computer
Engineering
ABSTRACT

A fast full-wave Method of Moments (MoM) solution of the governing integral equation is developed to analyze the problem of electromagnetic (EM) wave radiation and scattering from large finite arrays of printed elements in grounded multilayered media. The full-wave solver developed in this study utilizes a multilayered media dyadic Green’s function as the kernel of the governing integral equation. The Green’s function is efficiently evaluated via an asymptotic closed-form approximation combined with a fast numerical integration method, which significantly reduces the computational cost required to fill the MoM operator matrix. Various iterative solvers, which include a fast matrix-vector multiplication scheme using the fast Fourier transform (FFT), are used to solve the MoM matrix equation to reduce the storage requirements and solving time, and a discrete Fourier transform (DFT)-based preconditioner is also implemented to accelerate the convergence of iterative solvers. The latter approach is henceforth referred to as the preconditioned iterative (PI)-MoM. Additionally, the method is extended to handle arrays with non-rectangular element truncation boundaries. Furthermore, a hybrid uniform geometrical theory of diffraction (UTD)-MoM method is also developed as an alternative full-wave approach. This hybrid method uses a newly-introduced set of UTD-based global basis functions to drastically reduce the number of unknowns, and can thus eliminate the convergence problem of iterative solvers.
The full-wave solver in this work is implemented to predict radiation/scattering from arrays of four types of printed elements, namely printed dipoles, microstrip-line fed patch antennas, probe-fed patch antennas and aperture-coupled patch antennas, in grounded multilayered media. For the modeling of the coaxial of probe-fed patch antennas, a special attachment mode is included to enforce the continuity of the current at the probe-to-patch junction, as well as a magnetic frill generator is used to model the coaxial aperture. A detailed description of the PI-MoM and hybrid UTD-MoM method as well as several numerical results are included in this dissertation to validate the accuracy and utility of these methods.
This is dedicated to my family, friends, and fellow engineers
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VITA

January 30, 1968  . . . . . . . . . . . . . . . . . . . . . . . . . . . . Born - Bangkok, Thailand

1992  . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . B.S. Electrical Engineering, Kyoto University, Japan

1994  . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . M.Eng. Electrical Engineering, Kyoto University, Japan

1994-1999  . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . Government Service, Thammasat University, Bangkok, Thailand

2002  . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . M.S. Electrical Engineering, The Ohio State University, Columbus, OH

2002-present  . . . . . . . . . . . . . . . . . . . . . . . . . . . The Ohio State University, Columbus, OH

FIELDS OF STUDY

Major Field: Electrical Engineering

Studies in:
   Electromagnetics (Major area)
   Communications and Signal Processing
   Mathematics
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CHAPTER 1

INTRODUCTION

1.1 Scope and Motivation

Printed antennas form a class of low-profile antennas which include microstrip patch antennas, printed dipole antennas, printed slot antennas, etc.; their common feature is that they basically consist of four parts [1]:

1. a very thin flat metallic radiator or aperture

2. a dielectric substrate with or without dielectric superstrate

3. a ground plane

4. a feed.

They have found diverse applications in both commercial and military sectors mainly because of the following advantages[1, 2]:

- They are low-profile antennas.

- They are easily conformable to both planar and nonplanar surfaces, which along with their low-profile makes them suitable for aircraft applications.
• They are light-weight.

• They are easy and inexpensive to manufacture using printed-circuit technology.

• They can be easily integrated on the same substrate where the feed line and the source are embedded.

• They are mechanically robust when mounted to a rigid surface.

• They can be designed to produce a wide variety of patterns and polarizations, depending on the mode excited and the element shape.

However, printed antennas also have some disadvantages compared to other antennas including[3, 4]

• low efficiency and possible dielectric loss

• narrow bandwidth

• lower power handling capability

• detrimental effects of surface waves which can be excited.

The narrow bandwidth limitation may be desirable in some security systems, but broad bandwidths are typically more advantageous in most applications. Increasing substrate thickness can increase the bandwidth and efficiency, but it also potentially increases the surface wave power, which in turn degrades the antenna performance; thus, there is a limit on the substrate thickness. One technique to improve the bandwidth while maintaining the efficiency is to implement a substrate using more than one layer of properly-designed dielectric materials, which leads to the use of a multilayered medium.
In general, a single antenna element has a broad beamwidth and can provide only relatively low directive gain, especially in the case of printed antennas, thus it is not practical for applications which require higher gain such as long-distance communications or those demanding narrower beamwidth such as radar systems. Increasing the electrical size of the antenna can achieve higher gain and narrower beamwidth, but it is considered inefficient and costly. An alternative way to accomplish these goals without enlarging the antenna element is to construct an array of antenna elements, which forms a group of radiating elements assembled in such a way as to obtain a high directivity, or even a desired radiation pattern by adjusting the array parameters such as geometrical configuration, element spacing, excitation and so on. Exploiting the advantages of printed antennas and the versatility of antenna arrays, finite phased arrays of printed elements in multilayered media have found increasingly broader applications ranging from mobile communications to phased-array radar systems, because they can be built by means of monolithic fabrication and be easily integrated in the same medium where active devices such as phase shifters and amplifiers are embedded. Typically, printed antenna arrays can potentially provide desirable performances with relatively low cost, low profile with conformality and light weight.

As mentioned earlier, printed antennas generally have relatively narrow bandwidths and low power handling capabilities. To minimize these disadvantages and build an efficient antenna array of printed elements, accurate analysis techniques and optimum array design are necessary. It is therefore of great interest to develop an analysis tool for predicting the radiation and scattering from large finite arrays in grounded multilayered media as shown in Figure 1.1. An integral equation technique based on the method of moments (MoM) is one suitable numerical analysis approach for solving this type of problem. Since the number of unknowns is proportional to the number of array elements, when the array size
becomes very large, as with arrays in radar applications where a narrow beamwidth is required to obtain a high resolution, the conventional MoM can become highly inefficient or even intractable due to the computational cost and memory storage requirements for solving the MoM matrix equation. Moreover, there are some difficulties associated with the computation of the elements in the MoM operator matrix, which include the evaluation of Sommerfeld type integrals if the grounded multilayered media Green’s function is chosen as the kernel of the governing integral equation to be solved by MoM. Such a kernel is generally chosen since it restricts the unknowns to be solved to only the printed antenna elements, because it knows about the grounded multilayered medium. Some techniques and algorithms have been and are still being developed by others to reduce the computation time and also to overcome the memory storage limitations and other difficulties for solving large printed array problems.

![Array Element](image)

**Figure 1.1: A finite patch antenna array in a multilayered substrate**

In this research work, an asymptotic approximation technique combined with an efficient numerical integration method has been employed to efficiently evaluate the conventional Sommerfeld integral form of the Green’s functions pertaining to both electric current sources and magnetic current sources in a grounded multilayered dielectric medium, and
therefore significantly accelerate the evaluation of the MoM impedance (or operator) matrix elements as well as the MoM excitation vector elements pertaining to the integral equation formulation for the unknown array element currents.

Once a MoM matrix equation is obtained, it can be solved by a direct matrix solver if the number of unknowns is small. However, as mentioned previously, the number of unknowns increases as the number of array elements increases resulting directly in the increase in the size of the MoM matrix equation, and therefore making it difficult and highly inefficient to apply a direct solver due to limitations on computational resources. Although iterative matrix solvers can be used here to alleviate the problem, generally, the more the number of unknowns, the more the number of iterations required for the solutions to converge; hence, a proper preconditioner must thus be implemented to accelerate the convergence rate. In this work, a Discrete Fourier Transform(DFT)-based preconditioner has also been developed to be applied in various iterative matrix solvers, in order for this method to be able to handle very large array problems. Thus, this MoM together with the iterative approach with preconditioning will be referred to as the PI-MoM as also indicated in the abstract.

The method mentioned so far is based on the assumption that arrays of interest have rectangular element truncation boundaries, whose associated MoM operator matrix is block Toeplitz with Toeplitz blocks (BTTB) due to the periodicity in the array elements. This property is highly preferable because only elements in first few rows are required to be evaluated, while other elements can be constructed using these few rows. Thus both computational cost and memory storage can be saved. Moreover, a fast matrix-vector multiplication can be carried out in terms of a fast Fourier transform, which accelerates the iteration process. However, arrays in practical applications can have non-rectangular array
element truncation boundaries, such as circular, elliptical or hexagonal arrays; thus it is desirable to include the capability to handle these non-rectangular arrays in the full-wave solver developed in this work as well.

Although the preconditioned iterative solver can be effective for most MoM matrix equations, the convergence problem might still arise due to a large condition number of the MoM operator matrix. This problem typically occurs when either the total thickness of the multilayered medium becomes electrically large or the dielectric constant of the medium is large. Therefore, a hybrid UTD-MoM method is developed in this work to be used as an alternative solver, whose basic idea is that relating unknown current coefficients by a newly-introduced set of global UTD-based basis functions can significantly reduce the number of unknowns needed to be solved. Since the number of unknowns is small, a direct matrix solver can be used to solve the new matrix equation without significantly increasing the computational cost, thus it can help eliminate the convergence problem. This is highly advantageous because it can be implemented without changing MoM basis functions and it is expected to work well with any subsectional basis function.

The printed elements included in this work are printed dipoles and rectangular patch antennas, which are considered most fundamental elements among printed antennas. Patch antennas are also most wide-used printed elements due to their versatility. There are three commonly-used methods to feed this antenna, namely, via a microstrip-line feed, via a coaxial probe feed, and via aperture coupling, all of which will be dealt with in this work. Although patch antenna arrays with microstrip line feed network have an advantage in ease of fabrication, they can suffer from radiation degradation due to radiation from feed networks and certain constraints on element spacing due to the feed network geometry. Therefore, coaxial feeds are more preferable for array applications, because a feed network
can be put underneath a ground plane. Since the input impedance of coaxial-fed patch can be easily affected by the probe modeling, more accurate models of coaxial probe and excitation are required. In this work, a probe is represented by unknown vertical expansion modes together with a special attachment mode to enforce the continuity of current at a wire-to-patch junction. Also a magnetic frill generator, which is considered an accurate model for coaxial aperture, is used as an excitation for each of the probe feeds. This leads to an increase in the number of unknowns and complexity of the MoM matrix equation; consequently, the preconditioner as well as the iterative solvers have to be modified to compensate for these effects.

In summary, the main objective of this dissertation is to develop an efficient numerical MoM method for predicting the performance of large finite arrays of printed elements in grounded multilayered media; the scope of this dissertation covers:

- Develop an efficient approach to evaluate the multilayered media Green’s function for both electric and magnetic current sources.

- Implement a fast MoM-based full-wave solver to predict both radiation and scattering from large finite planar periodic arrays of printed elements in multilayered media by utilizing an efficient DFT-based preconditioner, leading to the PI-MoM approach, which speeds up the convergence process of iterative solvers.

- Develop a hybrid UTD-MoM method to be used as an alternative solver for the MoM matrix equation.
Include four basic types of printed antenna array elements, namely, the printed dipole, the microstrip-line fed rectangular patch antenna, the probe-fed rectangular patch antenna, and the aperture-coupled rectangular patch antenna. These are some of the most commonly used elements in printed antenna applications.

1.2 Historical Development

Since the concept of microstrip antennas was first proposed by Deschamps [5] in 1953, there has been an extensive amount of investigation and development on microstrip antenna analysis, especially via full-wave numerical analysis techniques. An analysis tool is useful because it can provide an insight into the operation of an antenna and thus aid the design process as well as the study of how each design parameter affects the antenna performance, and hence lead to information on how optimal performance can be achieved. Due to the complexity of the microstrip antenna structure which involves a dielectric inhomogeneity of the substrate, an accurate analytic solution is unavailable except for some approximate ones based on a cavity model or a transmission-line model [6–9]. Therefore, a numerical approach based on a rigorous full wave formulation is necessary in order to always obtain a reasonably accurate solution.

As mentioned earlier, MoM is the most widely-used numerical method for analyzing problems dealing with the radiation/scattering from microstrip printed antennas. Early work regarding application of MoM to single microstrip antenna analysis can be found in [10], where image theory was used together with modifications in MoM operator matrix to account for the presence of dielectric slab without using the microstrip Green’s function. Later work [11–16] makes use of the microstrip Green’s function which is represented in terms of a spectral domain integral to acquire more accurate antenna parameters such
as input impedance, mutual coupling and radar cross section. The work related to non-rectangular patches such as circular patches can be found in [17, 18].

It is worthwhile noting that in most of the early work, the feed is typically modeled as an equivalent vertical electric source and the unknowns are associated with only currents on the microstrip patch, which in some cases cannot give accurate input impedance results. Thus, there have been efforts to include a more accurate feed model into the MoM solutions. The work related to developing more accurate feed models for patch antennas can be found in [19–23], while wire-to-plate models for accurately modeling wire-plate junctions have been investigated in [24, 25], on which the attachment model used in this work is based.

That single microstrip antennas cannot provide sufficiently high directive gain necessitates the array configuration as discussed earlier. The MoM has thus been extended to analyze radiation/scattering from both infinite and finite arrays of microstrip antennas. Studies related to infinite arrays can be found in [21, 26–28], while finite arrays have been investigated in [29–32]. As the size of finite arrays becomes larger, the number of unknowns also increases resulting in the difficulty mentioned earlier. Numerous approaches have been developed to cope with this difficulty, which can be categorized into three groups, namely accelerating the evaluation of MoM matrix elements, developing a fast matrix solver, and limiting the number of unknowns. The first one is essentially related to the evaluation of the spectral integral in the microstrip Green’s function. In [33–36], an asymptotic extraction technique is developed to evaluating the Green’s function for small separations and a closed-form asymptotic representation is used to approximate the Green’s function for large separations, while in [37–39], a full-wave discrete complex image technique is applied to obtain a closed-form spatial domain Green’s function. To reduce the matrix solving time, several approaches have been developed such as the application of fast matrix-vector
multiplication using FFT [40, 41], the fast multipole method (FMM) [42, 43], and the development of preconditioners [44–49]. Finally, in [50–52], new sets of basis functions are developed based on high frequency techniques to drastically reduce the number of unknowns. Also, analysis of finite arrays using approximate high frequency techniques can be found in [53–55].

1.3 Outline of the Dissertation

In the earlier part of this introduction chapter, some background information regarding this research work and its scope were presented. Also mentioned is the historical development of the research in this area. A short overview of the remaining chapters is as follows:

Chapter 2 begins by presenting the relevant vector potentials in the spectral domain for an arbitrarily oriented point electric (or magnetic) current in the grounded multilayered medium and the spectral domain representations for the vector potentials, which are used to derive the Green’s functions pertaining to both electric and magnetic current sources. Then the details of derivations of the multilayered Green’s functions for the electric field due to both horizontal and vertical electric point sources as well as a magnetic current source on the ground plane will be given. It is noted that these Green’s functions have to expressed in proper forms such that their asymptotic approximations, or fast numerical integration methods are applicable for evaluating these Green’s functions approximately in closed form, or numerically, respectively.

Chapter 3 describes the efficient techniques used in this work to evaluate the multilayered media Green’s functions, namely, the numerical integration and the asymptotic closed-form approximation approaches, respectively. The details of the evaluation of each function associated with the multilayered media Green’s functions are described later in
appendix A. Also presented in this chapter is the pole extraction scheme which needs to be developed as part of the evaluation process.

Chapter 4 deals with the formulation of the electric field integral equation (EFIE) governing an array of printed elements in multilayered medium, and its MoM implementation. First, the integral equation as well as the excitation models are described followed by the descriptions of the basis functions used in this work. Then the MoM implementation using these basis functions will be presented and the elements of both MoM operator matrix and excitation vector will also be given to conclude this chapter.

Chapter 5 includes detailed descriptions of the fast matrix-vector multiplication and the DFT-based preconditioner, leading to the PI-MoM approach, which can be used in any iterative solver, such as conjugate gradient (CG), biconjugate gradient stabilized (BCGSTAB), generalized minimal residual (GMRES) and so on. It begins by discussing the BTTB property of the MoM operator matrix associated with finite planar periodic arrays which also have rectangular element truncation boundaries. The fast matrix-vector multiplication using this BTTB property will then be presented and the implementation of the DFT-based preconditioner developed in this work will be described in detail. Also given in this chapter are some numerical results for arrays of various printed elements in grounded multilayered media using the PI-MoM.

Chapter 6 describes the PI-MoM method used to analyze arrays with non-rectangular element truncation boundaries. First, the array shape matrix, which is used to represent the truncation boundary, will be presented followed by the application of the array shape matrix in the PI-MoM for non-rectangular array problems. Some numerical results of non-rectangular arrays are also presented based on this PI-MoM in this chapter.
Chapter 7 discusses the hybrid UTD-MoM method developed in this work as an alternative matrix solver. It begins by presenting the results from a DFT-based UTD ray analysis of large finite arrays, followed by the hybrid UTD-MoM approach which is based on the results from the UTD ray analysis. The implementation of this method for both rectangular and non-rectangular will be described and some numerical results obtained by this method will be presented in comparison to those from the PI-MoM method.

Chapter 8 contains general conclusions regarding this research work and concludes this dissertation with some recommendations for future work.

Appendix A includes the details of the numerical integrations for each function associated with the multilayered media Green’s functions. The important topics discussed here are the development of the large argument approximations and the resulting Green’s function integrals needed to be evaluated numerically.

Appendix B contains information on the postprocessing of the PI-MoM and UTD-MoM solutions in the code to obtain important array quantities such as radiation patterns, input impedance and radar cross section.

Appendix C discusses a generalized Thevenin’s theorem which can be used to combine a result from an antenna array full-wave analysis with a feed network.

An $e^{+j\omega t}$ time convention for the sources and fields is assumed and suppressed in the following development.
As mentioned earlier, an integral equation technique based on MoM is the numerical analysis approach which is utilized in this work for numerically solving the electromagnetic problems of the radiation/scattering from large finite arrays in grounded multilayered media. To apply the integral equation method to this type of a problem, the original problem is first converted to an equivalent problem via the field equivalence theorem[56], and the unknown quantities which are to be solved numerically via the MoM procedure are typically represented in terms of the equivalent currents. Therefore, an appropriate Green’s function is required to obtain the fields radiated by the currents. For problems in multilayered media, choosing the multilayered media dyadic Green’s function to be the kernel of the integral equation automatically restricts the unknown equivalent currents to reside only on the array elements; this immediately leads to a significant reduction in the number of unknowns needed to be solved. However, the total number of unknowns is typically larger than the total number of antenna elements present in the array.

In this chapter, the derivation of the multilayered media dyadic Green’s functions for the configuration of a multilayered material medium shown in figure 2.1 will be presented. As shown in the figure, $z_i$ denotes the $z$ coordinate of the bottom of the $i^{th}$ layer and $(\mu_i, \epsilon_i)$
denote the permeability and permittivity, respectively, of the $i^{th}$ layer. It is also noted that the $n^{th}$ layer is the half free space region.

\[ z_1 = 0, \quad z_2, \quad z_3, \quad \ldots, \quad z_{m-1}, \quad z_m, \quad z_{m+1}, \quad \ldots, \quad z_n \]

Figure 2.1: Configuration of a grounded multilayered medium

This chapter will first present the relevant vector potentials in the spectral domain for an arbitrarily oriented point electric (or magnetic) current in the grounded multilayered medium. Then the multilayered media dyadic Green’s functions pertaining to both horizontal and vertical electric sources and horizontal magnetic sources on the ground plane will be found directly from these potentials in the following sections.

### 2.1 Field Representation in Spectral Domain

The electric and magnetic fields $(E_i, H_i)$ in any $i^{th}$ layer of the configuration in figure 2.1 radiated by arbitrarily oriented electric or magnetic current point sources can be
obtained using two independent vector potentials[56], namely,

\[ \mathbf{A}_i = \hat{z}A_i, \]  

(2.1)

and

\[ \mathbf{F}_i = \hat{z}F_i, \]  

(2.2)

for the \( i \)th layer, via

\[ \mathbf{E}_i = -\nabla \times \mathbf{F}_i - j\omega \mu_i \mathbf{A}_i + \frac{1}{j\omega \epsilon_i} \nabla \nabla \cdot \mathbf{A}_i, \]  

(2.3)

\[ \mathbf{H}_i = \nabla \times \mathbf{A}_i - j\omega \epsilon_i \mathbf{F}_i + \frac{1}{j\omega \mu_i} \nabla \nabla \cdot \mathbf{F}_i. \]  

(2.4)

The fields due to \( \mathbf{A}_i \) and \( \mathbf{F}_i \) are referred to as TM to \( z \) and TE to \( z \), respectively. Both \( \mathbf{A}_i \) and \( \mathbf{F}_i \) satisfy the following wave equation in the region excluding the source:

\[ (\nabla^2 + k_i^2) \begin{bmatrix} \mathbf{A}_i \\ \mathbf{F}_i \end{bmatrix} = 0, \]  

(2.5)

where the wave number \( k_i = \omega \sqrt{\mu_i \epsilon_i} \). These two sets of potentials are decoupled except at the boundary where the current source exists.

In the plane wave spectral domain, \( \mathbf{A}_i \) and \( \mathbf{F}_i \) can be represented by

\[ \begin{bmatrix} \mathbf{A}_i \\ \mathbf{F}_i \end{bmatrix} = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \begin{bmatrix} a_i \\ f_i \end{bmatrix} e^{-jk_x x} e^{-jk_y y} dk_x dk_y, \]  

(2.6)

and \( a_i \) and \( f_i \) can be written as

\[ \begin{bmatrix} a_i \\ f_i \end{bmatrix} = \begin{bmatrix} C_i^m \\ C_i^e \end{bmatrix} \cos[\xi_i(z - z_i)] + \begin{bmatrix} B_i^m \\ B_i^e \end{bmatrix} jk_0 \frac{\sin[\xi_i(z - z_i)]}{\xi_i}, \]  

(2.7)

where \( \xi_i = \sqrt{k_i^2 - k_x^2 - k_y^2} \). Note that the superscripts \( m \) and \( e \) denote TM to \( z \) and TE to \( z \) fields, respectively. The unknown coefficients \( C_i^{m,e} \) and \( B_i^{m,e} \) can then be determined by applying appropriate boundary conditions for the electric and magnetic fields.
### 2.2 Field due to a Horizontal (or Tangential) Electric Current Source

In general, choosing a layer boundary to coincide with the location of the horizontal electric current source can make the analysis simpler. Here, a horizontal current source at the \( m \)th layer denotes the source located at the boundary between the \((m - 1)\)th and \( m \)th layer, or \( z = z_m \), as shown in figure 2.1. This notation will be used throughout this work. Now, assume that there exists a horizontal electric point current source at \( z = z_m \) given by

\[
J = p_t \delta(x - x')\delta(y - y')\delta(z - z_m) \triangleq J_s \delta(z - z_m), \tag{2.8}
\]

where \( p_t = \hat{x}_p x + \hat{y}_p y \). In the plane wave spectral domain, the \( J_s \) in (2.8) becomes

\[
J_s = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_t e^{-jkx(x-x')} e^{-jk_y(y-y')} dk_x dk_y, \tag{2.9}
\]

Applying the tangential magnetic field boundary conditions \( \hat{z} \times (H_m - H_{m-1}) = J_s \) at any \( z = z_m \) yields

\[
C^m_m - \left( C^m_{m-1} \cos \xi_{m-1} \Delta z_{m-1} + \frac{B^m_{m-1}}{\xi_{m-1}} jk_0 \sin \xi_{m-1} \Delta z_{m-1} \right) = \frac{k_x p_x + k_y p_y}{k_x^2 + k_y^2} j e^{j \psi}, \tag{2.10}
\]

\[
B^e_m = \frac{\mu_m}{\mu_{m-1}} \left( B^e_{m-1} \cos \xi_{m-1} \Delta z_{m-1} + \frac{C^e_{m-1}}{k_0} j \xi_{m-1} \sin \xi_{m-1} \Delta z_{m-1} \right)
= -\frac{\omega \mu_m}{k_0} \frac{k_x p_x + k_y p_y}{k_x^2 + k_y^2} j e^{j \psi}, \tag{2.11}
\]

where \( \psi = k_x x' + k_y y' \) and \( \Delta z_{i-1} = z_i - z_{i-1} \), i.e., the thickness of the \( i \)th layer. Likewise, applying the tangential electric field boundary condition \( \hat{z} \times (E_m - E_{m-1}) = 0 \) yields

\[
B^m_m - \frac{\epsilon_m}{\epsilon_{m-1}} \left( B^m_{m-1} \cos \xi_{m-1} \Delta z_{m-1} + \frac{C^m_{m-1}}{k_0} j \xi_{m-1} \sin \xi_{m-1} \Delta z_{m-1} \right) = 0, \tag{2.12}
\]

\[
C^e_m - \left( C^e_{m-1} \cos \xi_{m-1} \Delta z_{m-1} + \frac{B^e_{m-1}}{\xi_{m-1}} jk_0 \sin \xi_{m-1} \Delta z_{m-1} \right) = 0. \tag{2.13}
\]
By introducing the following two matrices $\mathbf{D}_i^m$ and $\mathbf{D}_i^e$ as

$$
\mathbf{D}_i^m = \begin{bmatrix}
\cos(\xi_i \Delta z_i) & j \frac{k_0}{\xi_i} \sin(\xi_i \Delta z_i) \\
-j \frac{\xi_i \xi_{i-1}}{\epsilon_{i-1} k_0} \sin(\xi_{i-1} \Delta z_{i-1}) & \frac{\epsilon_i}{\xi_i} \cos(\xi_{i-1} \Delta z_{i-1})
\end{bmatrix},
$$

(2.14)

and

$$
\mathbf{D}_i^e = \begin{bmatrix}
\cos(\xi_i \Delta z_i) & j \frac{k_0}{\mu_i} \sin(\xi_i \Delta z_i) \\
j \frac{\mu_i \xi_i \xi_{i-1}}{\mu_{i-1} k_0} \sin(\xi_{i-1} \Delta z_{i-1}) & \frac{\mu_i}{\mu_{i-1}} \cos(\xi_{i-1} \Delta z_{i-1})
\end{bmatrix},
$$

(2.15)

(2.10), (2.12) and (2.11), (2.13) can be rewritten in terms of matrix relations as

$$
\begin{bmatrix}
C_m^m & B_m^m \\
B_m^e & C_m^e
\end{bmatrix}
- \mathbf{D}_m^m
\begin{bmatrix}
C_{m-1}^m & B_{m-1}^m \\
B_{m-1}^e & C_{m-1}^e
\end{bmatrix}
= \begin{bmatrix}
k_x p_x + k_y p_y & k_x p_x + k_y p_y \\
0 & j e^{j \psi}
\end{bmatrix},
$$

(2.16)

$$
\begin{bmatrix}
C_m^e & B_m^e \\
B_m^m & C_m^m
\end{bmatrix}
- \mathbf{D}_m^e
\begin{bmatrix}
C_{m-1}^e & B_{m-1}^e \\
B_{m-1}^m & C_{m-1}^m
\end{bmatrix}
= \begin{bmatrix}
0 & \omega_{m,m} k_x p_x - k_y p_y \\
\omega_{m,m} k_x p_x + k_y p_y & j e^{j \psi}
\end{bmatrix}.
$$

(2.17)

Notice that boundary conditions at source-free boundaries, i.e., $z = z_i, i = 2, \ldots, m - 1$ and $i = m + 1, \ldots, n - 1$, are equal to the ones at $z = z_m$ without source terms. Hence, from (2.16) and (2.17), it can be easily found that

$$
\begin{bmatrix}
C_i^{m,e} \\
B_i^{m,e}
\end{bmatrix}
= \mathbf{D}_i^{m,e}
\begin{bmatrix}
C_{i-1}^{m,e} \\
B_{i-1}^{m,e}
\end{bmatrix}, \quad \text{for } i = 2, \ldots, m - 1.
$$

(2.18)

For the sake of convenience in solving for unknown coefficients, boundary conditions at $z = z_i, i = m + 1, \ldots, n - 1$ are given as

$$
\begin{bmatrix}
C_i^{m,e} \\
B_i^{m,e}
\end{bmatrix}
= \mathbf{U}_i^{m,e}
\begin{bmatrix}
C_{i+1}^{m,e} \\
B_{i+1}^{m,e}
\end{bmatrix}, \quad \text{for } i = m + 1, \ldots, n - 1,
$$

(2.19)

where

$$
\mathbf{U}_i^m = \begin{bmatrix}
\cos(\xi_i \Delta z_i) & -j \frac{\xi_i k_0}{\epsilon_{i+1} \xi_i} \sin(\xi_i \Delta z_i) \\
-j \frac{\xi_i \xi_{i+1}}{\epsilon_{i+1} k_0} \sin(\xi_{i+1} \Delta z_{i+1}) & \frac{\epsilon_i}{\epsilon_{i+1}} \cos(\xi_{i+1} \Delta z_{i+1})
\end{bmatrix},
$$

(2.20)

$$
\mathbf{U}_i^e = \begin{bmatrix}
\cos(\xi_i \Delta z_i) & -j \frac{\mu_{i,k} k_0}{\mu_{i+1} \xi_i} \sin(\xi_i \Delta z_i) \\
-j \frac{\mu_i \xi_i \xi_{i+1}}{\mu_{i+1} k_0} \sin(\xi_{i+1} \Delta z_{i+1}) & \frac{\mu_i}{\mu_{i+1}} \cos(\xi_{i+1} \Delta z_{i+1})
\end{bmatrix}.
$$

(2.21)

It can be easily shown that $\mathbf{U}_i^{m,e} = (\mathbf{D}_i^{m,e})^{-1}$. 

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Finally, applying the boundary condition at the ground plane, i.e., \( \hat{z} \times \mathbf{E}_1 = 0 \) and the radiation condition in the half space region yields

\[
B_1^m = C_1^e = 0,
\] (2.22)

and

\[
B_{n}^{m,e} = -\frac{\xi_n}{k_0} C_{n}^{m,e}, \quad \text{with } \text{Im} \, \xi_n \leq 0.
\] (2.23)

Using (2.18) and (2.22), one obtains

\[
\begin{bmatrix}
C_{m-1}^m \\
B_{m-1}^m
\end{bmatrix} = \mathbf{D}_{m-1}^m \mathbf{D}_{m-2}^m \cdots \mathbf{D}_{2}^m \begin{bmatrix}
C_1^m \\
B_1^m
\end{bmatrix}
\]

\[
= \mathbf{D}^{m,HE} \begin{bmatrix}
C_1^m \\
0
\end{bmatrix}
\]

\[
= C_1^m \begin{bmatrix}
\mathbf{D}_{11}^m \\
\mathbf{D}_{21}^m
\end{bmatrix},
\] (2.24)

and

\[
\begin{bmatrix}
C_{e-1}^e \\
B_{e-1}^e
\end{bmatrix} = \mathbf{D}_{e-1}^e \mathbf{D}_{e-2}^e \cdots \mathbf{D}_{2}^e \begin{bmatrix}
C_1^e \\
B_1^e
\end{bmatrix}
\]

\[
= \mathbf{D}^{e,HE} \begin{bmatrix}
0 \\
B_1^e
\end{bmatrix}
\]

\[
= B_1^e \begin{bmatrix}
\mathbf{D}_{12}^e \\
\mathbf{D}_{22}^e
\end{bmatrix}.
\] (2.25)

Note that if \( m - 1 = 1 \), then \( \mathbf{D}_{\{m,e\},HE} \) are identity matrices. Here, the \( HE \) in the superscripts denotes the parameters associated with the fields due to a horizontal electric current source in order to be distinguishable from those associated with other sources, which will be discussed later. Likewise, using (2.19) and (2.23) yields

\[
\begin{bmatrix}
C_{m}^{m,e} \\
B_{m}^{m,e}
\end{bmatrix} = \mathbf{U}_{m}^{m,e} \mathbf{U}_{m+1}^{m,e} \cdots \mathbf{U}_{n-1}^{m,e} \begin{bmatrix}
C_{n}^{m,e} \\
B_{n}^{m,e}
\end{bmatrix}
\]

\[
= \mathbf{U}^{\{m,e\},HE} \begin{bmatrix}
C_{n}^{m,e} \\
-\frac{\xi_n}{k_0} C_{n}^{m,e}
\end{bmatrix}
\]

\[
= C_{n}^{m,e} \begin{bmatrix}
\mathbf{U}_{11}^{\{m,e\},HE} \\
\mathbf{U}_{21}^{\{m,e\},HE}
\end{bmatrix},
\] (2.26)
where \( U_{m,e}^{HE} \) reduce to identity matrices if \( m = n \).

Now, using (2.24), (2.25) and (2.26) in (2.16) and (2.17), the unknown coefficients \( C_{m,e}^{1} \) and \( C_{n,e}^{m} \) can be found to be

\[
\begin{bmatrix}
C_{m,e}^{1} \\
C_{n,e}^{m}
\end{bmatrix}
= \begin{bmatrix}
C_{m,e}^{1} \\
C_{n,e}^{m}
\end{bmatrix} \frac{1}{\Delta_{m,e}^{HE}} \frac{k_{x}p_{x} + k_{y}p_{y} j e^{j\psi}}{k_{x}^{2} + k_{y}^{2}},
\] (2.27)

where

\[
C_{1,e}^{m'} = U_{21}^{m,HE} - \frac{\xi_{n}}{k_{0}} U_{22}^{m,HE},
\] (2.28)

\[
C_{n,e}^{m'} = \frac{\epsilon_{m}}{\epsilon_{m-1}} \left( D_{21}^{m,HE} \cos \xi_{m-1}\Delta z_{m-1} + \frac{\xi_{m-1}}{k_{0}} j D_{11}^{m,HE} \sin \xi_{m-1}\Delta z_{m-1} \right),
\] (2.29)

\[
\Delta_{m,e}^{HE} = \left( U_{11}^{m,HE} - \frac{\xi_{n}}{k_{0}} U_{12}^{m,HE} \right) C_{n,e}^{m'} - C_{1,e}^{m'} \left( D_{11}^{m,HE} \cos \xi_{m-1}\Delta z_{m-1} + \frac{\xi_{m-1}}{k_{0}} j D_{21}^{m,HE} \sin \xi_{m-1}\Delta z_{m-1} \right),
\] (2.30)

and

\[
\begin{bmatrix}
C_{n,e}^{m'} \\
B_{1,e}^{m'}
\end{bmatrix}
= \begin{bmatrix}
C_{n,e}^{m'} \\
B_{1,e}^{m'}
\end{bmatrix} \frac{1}{\Delta_{e}^{HE}} \frac{k_{y}p_{x} - k_{x}p_{y} j \mu_{m,e} e^{j\psi}}{k_{x}^{2} + k_{y}^{2} \mu_{m,e}^{2}},
\] (2.31)

where

\[
B_{1,e}^{m'} = U_{11}^{e,HE} - \frac{\xi_{n}}{k_{0}} U_{12}^{e,HE},
\] (2.32)

\[
C_{n,e}^{m'} = D_{12}^{e,HE} \cos \xi_{m-1}\Delta z_{m-1} + \frac{\xi_{m-1}}{k_{0}} j D_{22}^{e,HE} \sin \xi_{m-1}\Delta z_{m-1},
\] (2.33)

\[
\Delta_{e}^{HE} = \left( U_{21}^{e,HE} - \frac{\xi_{n}}{k_{0}} U_{22}^{e,HE} \right) C_{n,e}^{m'} - \frac{\mu_{m}}{\mu_{m-1}} D_{1}^{e',HE} \left( D_{22}^{e',HE} \cos \xi_{m-1}\Delta z_{m-1} + \frac{\xi_{m-1}}{k_{0}} j D_{12}^{e',HE} \sin \xi_{m-1}\Delta z_{m-1} \right).
\] (2.34)

It is noted that the coefficients \( C_{i}^{m',e',HE} \) and \( B_{i}^{m',e',HE} \) also satisfy (2.18) and (2.19), thus they are given by

\[
\begin{bmatrix}
C_{i}^{(m',e'),HE} \\
B_{i}^{(m',e'),HE}
\end{bmatrix}
= D_{i}^{m,e} \begin{bmatrix}
C_{i-1}^{(m',e'),HE} \\
B_{i-1}^{(m',e'),HE}
\end{bmatrix}, \quad \text{for } i = 2, \ldots, m-1,
\] (2.35)
and
\[
\begin{bmatrix}
C_{i}^{(m',e'),HE} \\
B_{i}^{(m',e'),HE}
\end{bmatrix} = U_{i}^{m,e} \begin{bmatrix}
C_{i+1}^{(m',e'),HE} \\
B_{i+1}^{(m',e'),HE}
\end{bmatrix}, \quad \text{for } i = m + 1, \ldots, n - 1, \tag{2.36}
\]

respectively.

Since the dyadic Green’s function \( \overline{G}_{i} \) relates to the electric field \( E_{i} \) and the point source \( p_{i} \) by
\[
E_{i} = \overline{G}_{i} \cdot p_{i}, \tag{2.37}
\]
the multilayered media dyadic Green’s function for the electric field due to an horizontal electric point source in the \( i^{th} \) layer can be identified by inspection from (2.3) as
\[
\overline{G}_{i}^{HE} (\bar{r}|\bar{r}') = \frac{1}{4\pi^{2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \overline{\Upsilon}_{i}^{HE} (k_{x}, k_{y}) e^{-jk_{x}(x-x')} e^{-jk_{y}(y-y')} dk_{x} dk_{y}, \tag{2.38}
\]
where
\[
\overline{\Upsilon}_{i}^{HE} (k_{x}, k_{y}) = \overline{\chi}_{e,HE}^{m} \overline{\gamma}_{i,t}^{m,HE} + \overline{\gamma}_{i,z}^{m,HE} + \overline{\chi}_{e,HE}^{e} \overline{\gamma}_{i}^{e,HE}, \tag{2.39}
\]
\[
\overline{\gamma}_{i,t}^{m,HE} (\xi_{i}) = \frac{k_{0}}{\omega \epsilon_{i}} \Delta_{m,HE} \left[ jC_{i}^{m',HE} \frac{\xi_{i}}{k_{0}} \sin \xi_{i}(z-z_{i}) + B_{i}^{m',HE} \cos \xi_{i}(z-z_{i}) \right], \tag{2.40}
\]
\[
\overline{\gamma}_{i,z}^{m,HE} (\xi_{i}) = \frac{1}{\omega \epsilon_{i}} \Delta_{m,HE} \left[ C_{i}^{m',HE} \cos \xi_{i}(z-z_{i}) + jB_{i}^{m',HE} \frac{k_{0}}{\xi_{i}} \sin \xi_{i}(z-z_{i}) \right], \tag{2.41}
\]
\[
\overline{\gamma}_{i}^{e,HE} (\xi_{i}) = \frac{\omega \mu_{m}}{k_{0}} \Delta_{e,HE} \left[ C_{i}^{e',HE} \cos \xi_{i}(z-z_{i}) + jB_{i}^{e',HE} \frac{k_{0}}{\xi_{i}} \sin \xi_{i}(z-z_{i}) \right], \tag{2.42}
\]
\[
\overline{\chi}_{e}^{m,HE} (k_{x}, k_{y}) = \frac{1}{k_{2}^{2} + k_{y}^{2}} (\hat{x} \hat{x} k_{x}^{2} + \hat{y} \hat{k}_{x} k_{y} + \hat{y} \hat{y} k_{y}^{2}), \tag{2.43}
\]
\[
\overline{\chi}_{e}^{m,HE} (k_{x}, k_{y}) = \hat{z} \hat{x} k_{x} + \hat{y} \hat{k}_{y}, \tag{2.44}
\]
\[
\overline{\chi}_{e}^{m,HE} (k_{x}, k_{y}) = \frac{1}{k_{2}^{2} + k_{y}^{2}} (\hat{x} \hat{x} k_{y}^{2} - \hat{y} \hat{k}_{x} k_{y} - \hat{y} \hat{y} k_{x}^{2}). \tag{2.45}
\]

In general, only the fields in the source layer, i.e., the \( m^{th} \) layer, and the fields in the half space region are of particular interest. It can be shown that the electric field at \( z = z_{m} \)
can be simplified to be

\[ E_m = \frac{\omega \mu_m}{4\pi^2 k_0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk_x dk_y e^{-jk_x(x-x')} e^{-jk_y(y-y')} \left\{ \left\langle \hat{x}'\hat{x} + \hat{y}'\hat{y} \right\rangle \frac{C_{m',HE}'}{\Delta_{m,HE}} e^{-jk_x(x-x')} e^{-jk_y(y-y')} \right\} \cdot \mathbf{p}_t. \]  

(2.46)

Now, introducing the following new functions:

\[ U^{HE} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{C_{m',HE}}{\Delta_{m,HE}} e^{-jk_x(x-x')} e^{-jk_y(y-y')} dk_x dk_y, \]  

(2.47)

\[ W^{HE} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{C_{m',HE}}{\Delta_{m,HE}} \left( \frac{k_0^2}{k_m^2} - \frac{B_{m',HE}}{\Delta_{m,HE}} \right) e^{-jk_x(x-x')} e^{-jk_y(y-y')} dk_x dk_y, \]  

(2.48)

then the tangential components of (2.46) can be rewritten as

\[ \hat{t}E_{m,t} = \frac{k_0 \omega \mu_m}{2\pi} \left[ \hat{x}' \left( U^{HE} + \frac{k_0^2}{k_m^2} \frac{\partial^2 W^{HE}}{\partial x^2} \right) + \hat{y}' \left( U^{HE} + \frac{k_0^2}{k_m^2} \frac{\partial^2 W^{HE}}{\partial y^2} \right) \right] \cdot \mathbf{p}_t. \]  

(2.49)

Defining \( \nabla_t = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} \) and \( \nabla_t' = \hat{x}' \frac{\partial}{\partial x'} + \hat{y}' \frac{\partial}{\partial y'} \), then the above equation can be rewritten in a more compact form as:

\[ \hat{t}E_{m,t} = \frac{k_0 \omega \mu_m}{2\pi} \left[ (\hat{x}' + \hat{y}') U^{HE} + \frac{k_0^2}{k_m^2} \nabla_t \nabla_t W^{HE} \right] \cdot \mathbf{p}_t. \]  

(2.50)

which can be easily applied in the mixed-potential integral equation (MPIE) formulation.

Now, the spectral integrals in (2.47) and (2.48) can be expressed in the Sommerfeld spectral representation form by using the following transformations given by:

\[ k_x = \lambda \cos \alpha, \]  

(2.51)
\[ k_y = \lambda \sin \alpha, \quad (2.52) \]
\[ x - x' = \rho \cos \phi, \quad (2.53) \]
\[ y - y' = \rho \sin \phi, \quad (2.54) \]

which will be referred to as the cylindrical coordinate transformation in later discussions.

Thus (2.47) and (2.48) become via (2.51)-(2.54) the following:

\[ U^{HE} = \frac{1}{2\pi} \int_0^\infty \int_0^{2\pi} \frac{1}{k_0^2} \Delta_{e,HE}^{m'} e^{-j\lambda \rho \cos(\alpha - \phi)} \lambda d\alpha d\lambda, \quad (2.55) \]
\[ W^{HE} = \frac{1}{2\pi} \int_0^\infty \int_0^{2\pi} \frac{1}{\lambda^2} \left( \frac{C_{m'}^{e,HE}}{\Delta_{e,HE}} - \frac{k_0^2}{k_m^2} \frac{B_{m'}^{m,HE}}{\Delta_{m,HE}} \right) e^{-j\lambda \rho \cos(\alpha - \phi)} \lambda d\alpha d\lambda. \quad (2.56) \]

Since
\[ J_0(\lambda \rho) = \frac{1}{2\pi} \int_0^{2\pi} e^{-j\lambda \rho \cos(\alpha - \phi)} d\alpha, \quad (2.57) \]

where \( J_0(\lambda \rho) \) is the Bessel function of zeroth order and \( \rho \) is the distance between the source and the observation point, (2.55) and (2.56) reduce to

\[ U^{HE} = \int_0^\infty \frac{C_{m'}^{e,HE}}{k_0^2} J_0(\lambda \rho) \lambda d\lambda, \quad (2.58) \]
\[ W^{HE} = \int_0^\infty \left( \frac{C_{m'}^{e,HE}}{\Delta_{e,HE}} - \frac{k_0^2}{k_m^2} \frac{B_{m'}^{m,HE}}{\Delta_{m,HE}} \right) J_0(\lambda \rho) \frac{1}{\lambda} d\lambda. \quad (2.59) \]

These \( U^{HE} \) and \( W^{HE} \) functions are the ones used in calculating the mutual coupling terms, or the elements of the MoM operator matrix, and their evaluation approaches will be discussed in the following chapter. It is noted that the second derivative of \( W^{HE} \) is not required in calculating mutual coupling, since the integral involving the derivative of \( W^{HE} \) can be modified such that the derivative of expansion function or testing function used in the MoM is needed instead.
2.3 Field due to a Vertical Electric Current Source in the Bottommost Layer

It can be shown that only the vector potential $\mathbf{A}$ is needed to find the field due to a vertical electric current source. In this section, the electric field due to a vertical source in the bottommost layer, i.e., the first layer, will be discussed. A vertical electric point current source in the first layer is given by

$$\mathbf{J} = z p_z \delta(x - x') \delta(y - y') \delta(z - z'),$$  \hspace{1cm} (2.60)

where $p_z$ denotes the vertical electric current source strength, and $z_1 \leq z' \leq z_2$. Now, let $\mathbf{A}_1 = z \mathbf{A}_1$ be the vector potential for the fields in the first layer, then $\mathbf{A}_1$ can be written as:

$$\mathbf{A}_1 = \mathbf{A}_r^1 + \mathbf{A}_s^1 \hspace{1cm} (2.61)$$

where $z \mathbf{A}_r^1$ is the vector potential for the fields due to the vertical source in an infinite homogeneous medium with the same electrical properties as the 1st layer, which can be written by

$$\mathbf{A}_r^1 = \frac{p_z e^{-jkz_R}}{4\pi R}, \hspace{1cm} (2.62)$$

where

$$R = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}.$$  

In the plane wave spectral domain, $\mathbf{A}_r^1$ can be represented by

$$\mathbf{A}_r^1 = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a_1^r e^{-jkx x} e^{-jky y} dk_x dk_y, \hspace{1cm} (2.63)$$

where $a_1^r$ is the Fourier transform of $\mathbf{A}_r^1$ given by

$$a_1^r = \frac{p_z}{j2\xi_1} e^{-j\xi_1|z-z'|} e^{j(k_x x'+k_y y')}. \hspace{1cm} (2.64)$$
Also, \( \hat{z}A_1^s \) is the vector potential due to the presence of the grounded multilayered structure and its Fourier transform can be given in the same form as (2.7), i.e.,

\[
a_1^s = C_1 \cos \xi_1 z + jB_1 \frac{k_0}{\xi_1} \sin \xi_1 z. \tag{2.65}
\]

Then, \( A_1 \) can be written in terms of spectral integral as

\[
A_1 = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a_1 e^{-jk_x x} e^{-jk_y y} dk_x dk_y, \tag{2.66}
\]

where

\[
a_1 = a_1^r + a_1^s. \tag{2.67}
\]

Likewise, \( A_i, i = 2, \ldots, n \) can be given by

\[
A_i = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a_i e^{-jk_x x} e^{-jk_y y} dk_x dk_y, \quad \text{for } i = 2, \ldots, n, \tag{2.68}
\]

where

\[
a_i = C_i \cos \xi_i (z - z_i) + \frac{k_0}{\xi_i} j \sin \xi_i (z - z_i). \tag{2.69}
\]

Applying the boundary condition \( \hat{z} \times \mathbf{E} = 0 \) at \( z = z_1 \) yields

\[
\frac{p_z}{2} e^{-j\xi_1 z'} e^{i\psi} + jk_0 B_1 = 0, \tag{2.70}
\]

where \( \psi = k_xx' + k_y y' \) as before. Likewise, from the boundary conditions at \( z = z_2 \), one obtains

\[
\begin{bmatrix} C_2 \\ B_2 \end{bmatrix} - \mathbf{D}_2^n \begin{bmatrix} C_1 \\ B_1 \end{bmatrix} = \frac{p_z}{2} e^{-j\xi_1 (z_2 - z')} e^{j\psi} \begin{bmatrix} \frac{1}{\xi_1} \\ \frac{\xi_2}{\xi_1 k_0} \end{bmatrix}, \tag{2.71}
\]

where the matrix \( \mathbf{D}_2^n \) is defined in (2.14). Finally, applying the boundary conditions at other boundaries, and the radiation condition in the half space region yields,

\[
\begin{bmatrix} C_2 \\ B_2 \end{bmatrix} = \mathbf{U}^m \mathbf{U}_3^m \cdots \mathbf{U}_{n-1}^m \begin{bmatrix} C_n \\ B_n \end{bmatrix}
\]

\[
= \mathbf{U}^{VE} \begin{bmatrix} C_n \\ -\frac{\xi_0}{k_0} C_n \end{bmatrix}
\]

\[
= C_n \begin{bmatrix} \mathbf{U}_1^{VE} - \frac{\xi_0}{k_0} \mathbf{U}_2^{VE} \\ \mathbf{U}_2^{VE} - \frac{\xi_0}{k_0} \mathbf{U}_2^{VE} \end{bmatrix}, \tag{2.72}
\]

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where $\mathbf{u}^{V_E1}$ reduces to the identity matrix if $n = 2$. Here, the $VE1$ in the superscript denotes the parameters associated with the fields due to a vertical electric current source located in the first layer.

Now, from (2.70), (2.71), and (2.72), the unknown coefficients can be found to be

$$
\begin{bmatrix}
C^{m,VE1}_n \\
C^{m,VE1}_1
\end{bmatrix}
= \begin{bmatrix}
C^{m',VE1}_n \\
C^{m',VE1}_1
\end{bmatrix}
\frac{1}{2\Delta^{m,VE1}} (-p_2 e^{j\psi}) ,
$$

(2.73)

where

$$
C^{m',VE1}_1 = \frac{1}{j\xi_1} \left( \mathbf{u}^{VE1}_{21} - \frac{\xi_n}{k_0} \mathbf{u}^{VE1}_{22} \right) \left( -e^{-j\xi_1(z_2 - z')} + j e^{-j\xi_1 z' \sin \xi_1 z_2} \right) - \frac{\epsilon_2}{j\xi_1 k_0} \left( \mathbf{u}^{VE1}_{11} - \frac{\xi_n}{k_0} \mathbf{u}^{VE1}_{12} \right) \left( e^{-j\xi_1(z_2 - z')} + e^{-j\xi_1 z' \cos \xi_1 z_2} \right) ,
$$

(2.74)

$$
C^{m',VE1}_n = -\frac{\epsilon_2}{\xi_1 k_0} \sin \xi_1 z_2 \left( -e^{-j\xi_1(z_2 - z')} + j e^{-j\xi_1 z' \sin \xi_1 z_2} \right) + \frac{\epsilon_2}{j\xi_1 k_0} \cos \xi_1 z_2 \left( e^{-j\xi_1(z_2 - z')} + e^{-j\xi_1 z' \cos \xi_1 z_2} \right) - \frac{\epsilon_2}{\xi_1 k_0} e^{j2 \cos \xi_1 z'} ,
$$

(2.75)

$$
\Delta^{m,VE1} = - \left( \mathbf{u}^{VE1}_{21} - \frac{\xi_n}{k_0} \mathbf{u}^{VE1}_{22} \right) \cos \xi_1 z_2 + \frac{\epsilon_2 \xi_1}{\xi_1 k_0} j \sin \xi_1 z_2 \left( \mathbf{u}^{VE1}_{11} - \frac{\xi_n}{k_0} \mathbf{u}^{VE1}_{12} \right) .
$$

(2.76)

The coefficients in other layers can be found from

$$
\begin{bmatrix}
C^{m',VE1}_1 \\
B_i^{m',VE1}
\end{bmatrix}
= \prod_{k=i}^{n-1} \mathbf{u}^{m'}_{k} \begin{bmatrix}
C^{m',VE1}_n \\
\frac{\xi_n}{k_0} C^{m',VE1}_n
\end{bmatrix}
= \mathbf{V}^{i,VE1} \begin{bmatrix}
C^{m',VE1}_n \\
\frac{\xi_n}{k_0} C^{m',VE1}_n
\end{bmatrix} .
$$

(2.77)

It is noted that $\mathbf{V}^{2,VE1} = \mathbf{u}^{VE1}$.

Using the above results in (2.67) and (2.69), and after some simplifications, $a_i, i = 1, \ldots , n$ can be given by

$$
a_1 = \frac{p_2 e^{j\psi}}{j2\xi_1 \Delta^{m,VE1}} \left\{ \left( \mathbf{u}^{VE1}_{11} - \frac{\xi_n}{k_0} \mathbf{u}^{VE1}_{12} \right) \frac{\epsilon_2 \xi_1}{\xi_1 k_0} \left[ \cos \xi_1 (z_2 - (z + z'')) + \cos \xi_1 (z_2 - |z - z'|) \right] \\
- j \left( \mathbf{u}^{VE1}_{21} - \frac{\xi_n}{k_0} \mathbf{u}^{VE1}_{22} \right) \left[ \sin \xi_1 (z_2 - (z + z'')) + \sin \xi_1 (z_2 - |z - z'|) \right] \right\} ,
$$

(2.78)
and

\[ a_i = -\frac{p_z e^{j\psi}}{2\Delta m, VE_1} \left[ C_{m', VE_1}^m \cos \xi_i(z - z_i) + \frac{j k_0}{\xi_i} B_{i, VE_1}^m \sin \xi_i(z - z_i) \right], \quad i = 2, \ldots, n, \]  

(2.79)

respectively. Now, the dyadic Green’s function relates to the electric field and the source by

\[ E_i = \overline{G} \cdot p_z. \]  

(2.80)

Thus, the multilayered media dyadic Green’s function representing the electric field in the \( i \)th layer for \( i \neq 1 \), which is due to a vertical electric current source in the bottom layer, can then be obtained by inspection as

\[ \overline{G}_i^{VE_1}(\vec{r} | \vec{r}') = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \overline{\Upsilon}_i^{VE_1}(k_x, k_y) e^{-jk_x(x-x')} e^{-jk_y(y-y')} dk_x dk_y, \]  

(2.81)

where

\[ \overline{\Upsilon}_i^{VE_1}(k_x, k_y) = \overline{\chi}_i^{VE_1} \Upsilon_i^{VE_1} + \overline{\chi}_i^{VE_1} \Upsilon_i^{VE_1}, \]  

(2.82)

\[ \Upsilon_i^{VE_1}(\xi_i) = -\frac{1}{2\omega \epsilon_i \Delta m, VE_1} \left[ C_{m', VE_1}^m \sin \xi_i(z - z_i) - \frac{j k_0}{\xi_i} B_{i, VE_1}^m \cos \xi_i(z - z_i) \right], \]  

(2.83)

\[ \Upsilon_i^{VE_1}(\xi_i) = -\frac{1}{j2\omega \epsilon_i \Delta m, VE_1} \left[ C_{m', VE_1}^m \cos \xi_i(z - z_i) + j B_{i, VE_1}^m k_0 \sin \xi_i(z - z_i) \right], \]  

(2.84)

\[ \overline{\chi}_i^{VE_1}(k_x, k_y) = \hat{x} \hat{z} \xi_i k_x + \hat{y} \hat{z} \xi_i k_y, \]  

(2.85)

\[ \overline{\chi}_i^{VE_1}(k_x, k_y) = \hat{z} (k_i^2 - \xi_i^2). \]  

(2.86)

For the electric field in the first layer, it is more convenient to express in the following form:

\[ E_1 = \hat{x} E_{1,x} + \hat{y} E_{1,y} + \hat{z} E_{1,z}, \]  

(2.87)
using the newly introduced functions \( V_{1a}^{VE1} \) and \( V_{1b}^{VE1} \) given by

\[
V_{1a}^{VE1} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk_x dk_y e^{-jk_x(x-x')} e^{-jk_y(y-y')} \frac{1}{2k_0\xi_1 \Delta m, VE1} \cdot \left[ \left( U_{11}^{VE1} - U_{12}^{VE1} \frac{\xi_n}{k_0} \right) \frac{\epsilon_2 \xi_1}{\epsilon_1 k_0} \cos \xi_1 (z_2 - |z - z'|) \right. \\
- j \left( U_{21}^{VE1} - U_{22}^{VE1} \frac{\xi_n}{k_0} \right) \sin \xi_1 (z_2 - |z - z'|) \right],
\]

(2.91)

\[
V_{1b}^{VE1} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk_x dk_y e^{-jk_x(x-x')} e^{-jk_y(y-y')} \frac{1}{2k_0\xi_1 \Delta m, VE1} \cdot \left[ \left( U_{11}^{VE1} - U_{12}^{VE1} \frac{\xi_n}{k_0} \right) \frac{\epsilon_2 \xi_1}{\epsilon_1 k_0} \cos \xi_1 (z_2 - (z + z')) \right. \\
- j \left( U_{21}^{VE1} - U_{22}^{VE1} \frac{\xi_n}{k_0} \right) \sin \xi_1 (z_2 - (z + z')) \right].
\]

(2.92)

Now, using the cylindrical coordinate transformation given in the previous section, and integrating over \( \alpha \) yields

\[
V_{1a}^{VE1} = \int_0^\infty d\lambda J_0(\lambda \rho) \lambda \frac{1}{2k_0\xi_1 \Delta m, VE1} \left[ \left( U_{11}^{VE1} - U_{12}^{VE1} \frac{\xi_n}{k_0} \right) \frac{\epsilon_2 \xi_1}{\epsilon_1 k_0} \cos \xi_1 (z_2 - |z - z'|) \right. \\
- j \left( U_{21}^{VE1} - U_{22}^{VE1} \frac{\xi_n}{k_0} \right) \sin \xi_1 (z_2 - |z - z'|) \right],
\]

(2.93)

\[
V_{1b}^{VE1} = \int_0^\infty d\lambda J_0(\lambda \rho) \lambda \frac{1}{2k_0\xi_1 \Delta m, VE1} \left[ \left( U_{11}^{VE1} - U_{12}^{VE1} \frac{\xi_n}{k_0} \right) \frac{\epsilon_2 \xi_1}{\epsilon_1 k_0} \cos \xi_1 (z_2 - (z + z')) \right. \\
- j \left( U_{21}^{VE1} - U_{22}^{VE1} \frac{\xi_n}{k_0} \right) \sin \xi_1 (z_2 - (z + z')) \right].
\]

(2.94)
It is noted that these $V_{V E1}^{V E1}$ and $V_{b V E1}^{V E1}$ functions are similar to $U^{HE}$ and $W^{HE}$ functions given in the previous section and can thus be evaluated in the same manner.

In a problem where a vertical probe extends beyond the first layer, the coupling between the electric field due to a vertical source in the first layer and a vertical current source in another layer might be needed. Therefore, it is also required to compute the normal component of the electric field in another layer as well. From (2.79), the vector potential for the fields in the $q^{th}$ layer, $A_q$, is given by $\hat{z}A_q$ where

$$A_q = -\frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk_x dk_y e^{-jk_x(x-x')} e^{-jk_y(y-y')}$$

$$\frac{p_z}{2\Delta m,VE1} \left[ C_{q'}^{m',VE1} \cos \xi_q(z-z_q) + \frac{jk_0}{\xi_q} B_{q'}^{m',VE1} \sin \xi_q(z-z_q) \right],$$

and the normal component, $E_{q,z}$, can then be given by

$$E_{q,z} = -j\frac{\omega\mu_q}{k^2_q} (k^2_q + \frac{\partial^2}{\partial z^2}) A_q.$$  

From (2.75) and (2.77), $C_{q}^{m',VE1}$ and $B_{q}^{m',VE1}$ are given by

$$\begin{bmatrix} C_{q}^{m',VE1} \\ B_{q}^{m',VE1} \end{bmatrix} = \mathbf{V}_{q}^{V E1} \begin{bmatrix} C_{n}^{m',VE1} \\ -\frac{\xi_n}{\xi_q} C_{n}^{m',VE1} \end{bmatrix}$$

and

$$\frac{\epsilon_2}{\epsilon_1 k_0} j^2 \cos \xi_{1'} z' \begin{bmatrix} \mathbf{V}_{q}^{V E1} - \frac{\xi_n}{\xi_q} \mathbf{V}_{q}^{V E1} \\ \mathbf{V}_{q}^{V E1} - \frac{\xi_n}{\xi_q} \mathbf{V}_{q}^{V E1} \end{bmatrix}.$$  

Now, defining new functions $V_{V E1}^{q a}$ and $V_{V E1}^{q b}$ as:

$$V_{V E1}^{q a} = -\frac{j}{2\pi k_0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk_x dk_y e^{-jk_x(x-x')} e^{-jk_y(y-y')}$$

$$\frac{1}{2\Delta m,VE1} \left[ C_{q}^{m',VE1} \cos \xi_q(z-z_q) + \frac{jk_0}{\xi_q} B_{q}^{m',VE1} \sin \xi_q(z-z_q) \right],$$

and,

$$V_{V E1}^{q b} = \frac{1}{2\pi k_0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk_x dk_y e^{-jk_x(x-x')} e^{-jk_y(y-y')}$$

$$\frac{\epsilon_2}{\epsilon_1 k_0} \sin \xi_{1'} z' \frac{1}{\Delta m,VE1}$$

$$\begin{bmatrix} \mathbf{V}_{q}^{V E1} - \frac{\xi_n}{\xi_q} \mathbf{V}_{q}^{V E1} \\ \mathbf{V}_{q}^{V E1} - \frac{\xi_n}{\xi_q} \mathbf{V}_{q}^{V E1} \end{bmatrix} \cos \xi_q(z-z_q),$$

(2.99)
respectively, then (2.96) can be rewritten as:

\[ \hat{E}_{q,z} = -\hat{E}_{q,z} - \hat{z} \frac{k_0 \omega \mu_q}{2\pi k_q^2} (k_q^2 V_{q,a}^{VE1} - \frac{\partial^2 V_{q,b}^{VE1}}{\partial z \partial z'}) \cdot \mathbf{p}_z. \]  

(2.100)

Now, applying the cylindrical coordinate transformation given in the previous section to (2.98) and (2.99), and integrating over \( \alpha \) yields

\[ V_{qa}^{VE1} = -\int_0^\infty \frac{j}{2k_0 \Delta m,VE1} \left[ C_{q,VE1}^{m',VE1} \cos \xi_q (z - z_q) + \frac{j k_0}{\xi_q} B_{q,VE1}^{m',VE1} \sin \xi_q (z - z_q) \right] J_0(\lambda \rho) \lambda d\lambda, \]  

(2.101)

and

\[ V_{qb}^{VE1} = -\int_0^\infty \frac{\epsilon_2 \sin \xi_1 z'}{\epsilon_1 \xi_1} \frac{j}{k_0 \Delta m,VE1} J_0(\lambda \rho) \lambda \left[ (V_{q,VE1}^{11} - \frac{\xi_n}{k_0} V_{q,VE1}^{12}) \frac{\xi_q}{k_0} j \sin \xi_q (z - z_q) + (V_{q,VE1}^{21} - \frac{\xi_n}{k_0} V_{q,VE1}^{22}) \cos \xi_q (z - z_q) \right] d\lambda, \]  

(2.102)

which can be evaluated efficiently.

Of another interest in this case is the tangential component of the electric field at \( z = z_m \), which is used to compute the coupling between the electric field due to a vertical source in the first layer and a horizontal source in the \( m^{th} \). It can be found from (2.81) to be

\[ \hat{E}_{m,t} = \frac{1}{4\pi^2 \omega \epsilon_m} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk_x dk_y e^{-j k_x (x-x')} e^{-j k_y (y-y')} j k_0 \frac{B_{m',VE1}^{m',VE1}}{2\Delta m,VE1} (\hat{x} k_x + \hat{y} k_y) \cdot \mathbf{p}_z. \]  

(2.103)

\( B_{m}^{m',VE1} \) can be found from (2.77) as

\[ B_{m}^{m',VE1} = \begin{pmatrix} V_{m,VE1}^{m',VE1} - \frac{\xi_n}{k_0} V_{m,VE1}^{m',VE1} \\ k_0 \end{pmatrix} C_{m}^{m',VE1} \]  

(2.104)
where $\mathcal{V}^{m,VE_1} = \prod_{k=m}^{n-1} \mathcal{U}_k^m$. Thus,

$$\hat{E}_{m,t} = \frac{1}{4\pi^2\omega\epsilon_m} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk_x dk_y e^{-jk_x(x-x')} e^{-jk_y(y-y')} \left( \mathcal{V}_{21}^{m,VE_1} - \frac{\xi_n}{k_0} \mathcal{V}_{22}^{m,VE_1} \right) \frac{\epsilon_2 \cos \xi_1 z'}{\epsilon_1 \Delta_m,VE_1} (\hat{x} k_x + \hat{y} k_y) \cdot \mathbf{p}.$$  \hspace{1cm} (2.105)

Defining the following function as follows

$$W^{VE_1} = \frac{1}{2\pi k_0^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \mathcal{V}_{21}^{m,VE_1} - \frac{\xi_n}{k_0} \mathcal{V}_{22}^{m,VE_1} \right) \frac{\epsilon_2 k_0 \sin \xi_1 z'}{j\epsilon_1 \xi_1 \Delta_m,VE_1} e^{-jk_x(x-x')} e^{-jk_y(y-y')} dk_x dk_y,$$  \hspace{1cm} (2.106)

then (2.105) can be rewritten using (2.106) as

$$\hat{E}_{m,t} = \frac{\omega \mu_0 \epsilon_0}{2\pi \epsilon_m k_0} \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) \frac{\partial W^{VE_1}}{\partial z'} \cdot \mathbf{p}.$$  \hspace{1cm} (2.107)

By using the cylindrical coordinate transformation and integrating over $\alpha$, (2.106) becomes

$$W^{VE_1} = \frac{1}{k_0^2} \int_0^{\infty} \left( \mathcal{V}_{21}^{m,VE_1} - \frac{\xi_n}{k_0} \mathcal{V}_{22}^{m,VE_1} \right) \frac{\epsilon_2 k_0 \sin \xi_1 z'}{j\epsilon_1 \xi_1 \Delta_m,VE_1} J_0(\lambda \rho) \lambda d\lambda,$$  \hspace{1cm} (2.108)

which is the form to be used in calculating the mutual coupling.

### 2.4 Field due to a Vertical Electric Current Source which is not in the Bottommost Layer

In this section, the fields due to a vertical electrical current source in a layer other than the bottommost layer will be considered. The approach used to find the fields due to a vertical electric current source in the bottommost layer, which is described in the previous section, can also be applied here. A vertical electric point current source in the $p^{th}$ layer is given by

$$\mathbf{J} = \hat{z} p_z \delta(x - x') \delta(y - y') \delta(z - z'),$$  \hspace{1cm} (2.109)
where $p_z$ denotes the vertical electric current source strength and $z_p \leq z' \leq z_{p+1}$. Let $A_p = \hat{z}A_p, p \neq 1$ be the vector potential for the fields in the $p^{th}$ layer, where a vertical current source is located, then $A_p$ can be written as:

$$A_p = A_r^p + A_s^p$$ (2.110)

where $\hat{z}A_r^p$ is the vector potential for the fields due to the vertical source on infinite homogeneous medium with the same electrical properties as the $p^{th}$ layer. Thus,

$$A_r^p = \frac{p_se^{-jk_pR}}{4\pi R},$$ (2.111)

where $p_z$ is the vertical electric current source strength and

$$R = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}.$$ 

In the spectral domain, $A_r^p$ can be represented by

$$A_r^p = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a_r^p e^{-j\xi x} e^{-j\xi y}dk_xdk_y, $$ (2.112)

where $a_r^p$ is the Fourier transform of $A_r^p$ given by

$$a_r^p = \frac{p_z}{j2\xi_p} e^{-j\xi_p z} e^{i(k_x x' + k_y y')}.$$ (2.113)

Also, $\hat{z}A_s^p$ is the vector potential due to the effects of scattering in the $p^{th}$ layer by the multilayered structure and its Fourier transform can be given in the same form as (2.7), i.e.,

$$a_s^p = C_p \cos \xi_p z + jB_p \frac{k_0}{\xi_p} \sin \xi_p z.$$ (2.114)

Then $A_p$ can be written in terms of spectral integral as

$$A_p = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a_p e^{-j\xi x} e^{-j\xi y}dk_xdk_y, $$ (2.115)
where
\[ a_p = a_p' + a_p^s. \]  
(2.116)

Likewise, \( A_i, i = 1, \ldots, n, i \neq p \) can be given by
\[ A_i = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a_i e^{-jk_x x} e^{-jk_y y} dk_x dk_y, \quad \text{for } i = 2, \ldots, n, i \neq p, \]  
(2.117)

where
\[ a_i = C_i \cos \xi_i (z - z_i) + \frac{k_0}{\xi_i} j \sin \xi_i (z - z_i). \]  
(2.118)

Applying the boundary condition \( \hat{z} \times E = 0 \) and \( \hat{z} \times H = 0 \) at \( z = z_p \) yields
\[ \begin{bmatrix} C_p \\ B_p \end{bmatrix} - \mathcal{D}_p m \begin{bmatrix} C_{p-1} \\ B_{p-1} \end{bmatrix} = \frac{p_z e^{-j\xi_p (z' - z_p)} e^{j\psi}}{2j} \begin{bmatrix} \frac{1}{\xi_p} \\ \epsilon_p k_0 \end{bmatrix}, \]  
(2.119)

where \( \psi = k_x x' + k_y y' \) as before. Likewise, from the boundary conditions at \( z = z_{p+1} \), one obtains
\[ \begin{bmatrix} C_{p+1} \\ B_{p+1} \end{bmatrix} - \mathcal{D}_{p+1} m \begin{bmatrix} C_p \\ B_p \end{bmatrix} = \frac{p_z e^{-j\xi_p (z_{p+1} - z')} e^{j\psi}}{2j} \begin{bmatrix} \frac{1}{\xi_p} \\ \epsilon_{p+1} \epsilon_p k_0 \end{bmatrix}, \]  
(2.120)

where the matrices \( \mathcal{D}_p m \) and \( \mathcal{D}_{p+1} m \) are defined in (2.14). Equation (2.120) can then be rewritten as
\[ \begin{bmatrix} C_p \\ B_p \end{bmatrix} = (\mathcal{D}_{p+1} m)^{-1} \left\{ \begin{bmatrix} C_{p+1} \\ B_{p+1} \end{bmatrix} - \frac{p_z e^{-j\xi_p (z_{p+1} - z')} e^{j\psi}}{2j} \begin{bmatrix} \frac{1}{\xi_p} \\ \epsilon_{p+1} \epsilon_p k_0 \end{bmatrix} \right\} \]
\[ = \mathcal{U}_p m \begin{bmatrix} C_{p+1} \\ B_{p+1} \end{bmatrix} - \frac{p_z e^{-j\xi_p (z_{p+1} - z')} e^{j\psi}}{2j} \begin{bmatrix} \frac{1}{\xi_p} \\ \epsilon_{p+1} \epsilon_p k_0 \end{bmatrix}, \]  
(2.121)

since \( \mathcal{U}_p m = (\mathcal{D}_{p+1} m)^{-1} \).

Next, applying the boundary conditions at the other boundaries yields
\[ \begin{bmatrix} C_{p-1} \\ B_{p-1} \end{bmatrix} = \mathcal{D}_{p-1} m \mathcal{D}_{p-2} m \cdots \mathcal{D}_2 m \begin{bmatrix} C_1 \\ B_1 \end{bmatrix} \]
\[ = \prod_{i=p-1}^2 \mathcal{D}_i m \begin{bmatrix} C_1 \\ B_1 \end{bmatrix}, \]  
(2.122)
and
\[
\begin{bmatrix} C_{p+1} \\ B_{p+1} \end{bmatrix} = \begin{bmatrix} \mathbf{u}^m_{p+1} & \mathbf{u}^m_{p+2} & \cdots & \mathbf{u}^m_{n-1} \end{bmatrix} \begin{bmatrix} C_n \\ B_n \end{bmatrix} = \prod_{i=p+1}^{n-1} \mathbf{u}^m_i \begin{bmatrix} C_n \\ B_n \end{bmatrix}. \tag{2.123}
\]

Finally, applying the boundary condition at the ground plane, i.e., \( \hat{z} \times \mathbf{E}_1 \mid_{z=0} = 0 \) and the radiation condition in the half space region yields
\[
B_1 = 0, \tag{2.124}
\]
and
\[
B_n = -\frac{\xi_n}{k_0} C_n, \quad \text{with } \text{Im } \xi_n \leq 0. \tag{2.125}
\]

Therefore, substituting (2.124) and (2.125) in (2.122) and (2.123), respectively, and using them in (2.119) and (2.121) yields
\[
\begin{bmatrix} C_p \\ B_p \end{bmatrix} = D^{Vep} \begin{bmatrix} C_1 \\ 0 \end{bmatrix} - \frac{p_z e^{-j\xi_p(z'-z_p)} e^{j\psi}}{2j} \begin{bmatrix} \frac{1}{\xi_p} \\ \frac{1}{k_0} \end{bmatrix}, \tag{2.126}
\]
and
\[
\begin{bmatrix} C_p \\ B_p \end{bmatrix} = U^{Vep} \begin{bmatrix} C_n \\ -\frac{\xi_n}{k_0} C_n \end{bmatrix} - \frac{p_z e^{j\xi_p(z'-z_p)} e^{j\psi}}{2j} \begin{bmatrix} \frac{1}{\xi_p} \\ \frac{1}{k_0} \end{bmatrix}, \tag{2.127}
\]
where \( D^{Vep} = \prod_{i=p}^2 D_i^m \), and \( U^{Vep} = \prod_{i=p}^{n-1} U_i^m \). Here, the \( Vep \) in the superscript denotes the parameters associated with the fields due to a vertical electric current source in the \( p^{th} \) layer.

Thus, from (2.126) and (2.127), one obtains
\[
\begin{align*}
U^{Vep} \begin{bmatrix} C_n \\ -\frac{\xi_n}{k_0} C_n \end{bmatrix} - D^{Vep} \begin{bmatrix} C_1 \\ 0 \end{bmatrix} &= \frac{p_z e^{j\xi_p(z'-z_p)} e^{j\psi}}{2j} \begin{bmatrix} \frac{1}{\xi_p} \\ \frac{1}{k_0} \end{bmatrix} - \frac{p_z e^{-j\xi_p(z'-z_p)} e^{j\psi}}{2j} \begin{bmatrix} \frac{1}{\xi_p} \\ \frac{1}{k_0} \end{bmatrix} \\
&= p_z e^{j\psi} \begin{bmatrix} \frac{1}{\xi_p} \sin \xi_p (z' - z_p) \\ \frac{1}{k_0} \cos \xi_p (z' - z_p) \end{bmatrix}.
\end{align*} \tag{2.128}
\]
Solving (2.128) for \( C_n \) and \( C_1 \) yields
\[
\begin{bmatrix} C_n^{m,VEp} \\ C_1^{m,VEp} \end{bmatrix} = \begin{bmatrix} C_1^{m',VEp} \\ C_n^{m',VEp} \end{bmatrix} \frac{p_2 e^{i \psi}}{\Delta_{m,VEp}},
\]  
(2.129)
where
\[
C_n^{m',VEp} = \frac{D_{21}^{VEp}}{\xi_p} \sin \xi_p (z' - z_p) - \frac{D_{11}^{VEp}}{k_0} j \cos \xi_p (z' - z_p),
\]  
(2.130)
\[
C_1^{m',VEp} = \frac{U_{21}^{VEp} - \xi_p U_{22}^{VEp}}{\xi_p} \sin \xi_p (z' - z_p) - \frac{U_{11}^{VEp} - \xi_p U_{12}^{VEp}}{k_0} j \cos \xi_p (z' - z_p),
\]  
(2.131)
and
\[
\Delta_{m,VEp} = \frac{D_{21}^{VEp}}{\xi_p} \left( U_{21}^{VEp} - \frac{\xi_n}{k_0} U_{22}^{VEp} \right) - \frac{D_{11}^{VEp}}{k_0} \left( U_{11}^{VEp} - \frac{\xi_n}{k_0} U_{12}^{VEp} \right).
\]  
(2.132)

The coefficients in other layers can be found from \( C_1^{m',VEp} \) and \( C_n^{m,VEp} \) as
\[
\begin{bmatrix} C_{p}^{m',VEp} \\ B_{p}^{m',VEp} \end{bmatrix} = D_{m,VEp} \begin{bmatrix} C_{1}^{m',VEp} \\ 0 \end{bmatrix} - \frac{p_2 e^{-j \xi_p (z' - z_p)} e^{i \psi}}{2 j} \begin{bmatrix} 1 \\ \frac{1}{k_0} \end{bmatrix},
\]  
(2.133)
\[
\begin{bmatrix} C_{i}^{m',VEp} \\ B_{i}^{m',VEp} \end{bmatrix} = \prod_{k=i}^{n-1} U_k^{m} \begin{bmatrix} C_{n}^{m',VEp} \\ -\frac{\xi_n}{k_0} C_{n}^{m',VEp} \end{bmatrix} = \nu_{i,VEp} \begin{bmatrix} C_{n}^{m',VEp} \\ -\frac{\xi_n}{k_0} C_{n}^{m',VEp} \end{bmatrix} \text{ for } i = p + 1, \ldots, n - 1,
\]  
(2.134)
\[
\begin{bmatrix} C_{i}^{m',VEp} \\ B_{i}^{m',VEp} \end{bmatrix} = \prod_{k=p+1}^{2} D_k^{m} \begin{bmatrix} C_{1}^{m',VEp} \\ 0 \end{bmatrix} = \nu_{i,VEp} \begin{bmatrix} C_{1}^{m',VEp} \\ 0 \end{bmatrix} \text{ for } i = 2, \ldots, p - 1,
\]  
(2.135)
and
\[
\begin{bmatrix} C_{i}^{m,VEp} \\ B_{i}^{m,VEp} \end{bmatrix} = \begin{bmatrix} C_{i}^{m',VEp} \\ B_{i}^{m',VEp} \end{bmatrix} \frac{p_2 e^{i \psi}}{D_{m,VEp}}.
\]  
(2.136)

Hence, \( a_i \) for each layer can then be given by
\[
a_i = \begin{cases} 
\frac{p_2 e^{i \psi}}{\Delta_{m,VEp}} \left[ \frac{\Delta_{m,VEp}}{\xi_p} e^{-j \xi_p |z' - z|} + C_p^{m',VEp} \cos \xi_p (z - z_p) + \frac{j k_0}{\xi_p} B_p^{m',VEp} \sin \xi_p (z - z_p) \right] & i = p \\
\frac{p_2 e^{i \psi}}{\Delta_{m,VEp}} \left[ C_i^{m',VEp} \cos \xi (z - z_i) + \frac{i k_0}{\xi_i} B_i^{m',VEp} \sin \xi_i (z - z_i) \right], & \text{otherwise}
\end{cases}
\]  
(2.137)
Now, let the dyadic Green’s function relate to the electric field and the source as

$$E_i = \overline{G_i} \cdot p_z,$$  \hspace{1cm} \text{(2.138)}

then the multilayered media dyadic Green’s function representing the electric field in the

$i^{th}$ layer for $i \neq p$, which is due to the vertical electric current source in the $p^{th}$ layer, can

then be given by

$$\overline{G_i}^{VEp} (\bar{r}|\bar{r}') = -\frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \overline{Y_i}^{VEp} (k_x, k_y)e^{-jk_i(x-x')}e^{-jk_p(y-y')}dk_xdk_y,$$  \hspace{1cm} \text{(2.139)}

where

$$\overline{Y_i}^{VEp} (k_x, k_y) = \overline{Y_i}^{VEp} (\xi_i) = \frac{1}{\omega \epsilon_i \Delta m_{VEp}} \left[ C_i^{m',VEp} \sin \xi_i (z - z_i) - j \frac{k_0}{\xi_i} D_i^{m',VEp} \cos \xi_i (z - z_i) \right],$$  \hspace{1cm} \text{(2.140)}

$$\overline{Y_i}^{VEp} (\xi_i) = \frac{1}{j \omega \epsilon_i \Delta m_{VEp}} \left[ C_i^{m',VEp} \cos \xi_i (z - z_i) + j B_i^{m',VEp} \frac{k_0}{\xi_i} \sin \xi_i (z - z_i) \right],$$  \hspace{1cm} \text{(2.141)}

$$\overline{\rho_i}^{VEp} (k_x, k_y) = \hat{x} \hat{z} \xi_i k_x + \hat{y} \hat{z} \xi_i k_y,$$  \hspace{1cm} \text{(2.142)}

$$\overline{\rho_i}^{VEp} (k_x, k_y) = \hat{z} \hat{z} (k_i^2 - \xi_i^2).$$  \hspace{1cm} \text{(2.143)}

For the electric field in the $p^{th}$ layer, it is more convenient to first simplify $\alpha_p$ given in

(2.137). Introducing a new matrix $\mathcal{W}^{VEp}$ given by

$$\mathcal{W}^{VEp} = \prod_{k=p+1}^{n-1} U_k^m,$$  \hspace{1cm} \text{(2.145)}

then it follows that

$$\mathcal{U}^{VEp} = \mathcal{U}_p^m \mathcal{W}^{VEp} =$$

$$\begin{bmatrix}
W_{11}^{VEp} \cos \xi_p \Delta z_p - \frac{\epsilon_p}{\epsilon_{p+1}} \xi_p \sin \xi_p \Delta z_p & W_{12}^{VEp} \cos \xi_p \Delta z_p - \frac{\epsilon_p}{\epsilon_{p+1}} \xi_p \sin \xi_p \Delta z_p \\
\frac{\xi_p}{k_0} W_{11}^{VEp} j \sin \xi_p \Delta z_p + \frac{\epsilon_p}{\epsilon_{p+1}} W_{21}^{VEp} \cos \xi_p \Delta z_p & \frac{\xi_p}{k_0} W_{22}^{VEp} j \sin \xi_p \Delta z_p + \frac{\epsilon_p}{\epsilon_{p+1}} W_{22}^{VEp} \cos \xi_p \Delta z_p
\end{bmatrix}.$$  \hspace{1cm} \text{(2.146)}
Using the above equation in (2.131) and (2.132), $C^{m,VEp}_p$ and $B^{m,VEp}_p$ can be found from (2.133). Substituting $C^{m,VEp}_p$ and $B^{m,VEp}_p$ in (2.137), and after some simplifications, $a_p$ can be given by

$$a_p = \frac{p_x e^{j\psi}}{j2\xi_p \Delta m,VEp} \left\{ \frac{\xi_p}{k_0} \mathcal{D}^{VEp}_{11} \mathcal{W}'_1 \left[ \cos \xi_p (\Sigma z_p - (z + z')) + \cos \xi_p (\Delta z_p - |z - z'|) \right] \right.$$ 

$$+ \frac{\epsilon_p k_0}{\epsilon_{p+1} \xi_p} \mathcal{D}^{VEp}_{21} \mathcal{W}'_2 \left[ \cos \xi_p (\Sigma z_p - (z + z')) - \cos \xi_p (\Delta z_p - |z - z'|) \right]$$

$$- \frac{\epsilon_p}{\epsilon_{p+1}} \mathcal{D}^{VEp}_{11} \mathcal{W}'_{2j} \left[ \sin \xi_p (\Sigma z_p - (z + z')) + \sin \xi_p (\Delta z_p - |z - z'|) \right]$$

$$- \mathcal{D}^{VEp}_{21} \mathcal{W}'_{1j} \left[ \sin \xi_p (\Sigma z_p - (z + z')) - \sin \xi_p (\Delta z_p - |z - z'|) \right]\right\}, \tag{2.147}$$

where

$$\mathcal{W}'_1 = \mathcal{W}^{VEp}_{11} - \frac{\xi_p}{k_0} \mathcal{W}^{VEp}_{12}, \tag{2.148}$$

$$\mathcal{W}'_2 = \mathcal{W}^{VEp}_{21} - \frac{\xi_p}{k_0} \mathcal{W}^{VEp}_{22}, \tag{2.149}$$

and

$$\Sigma z_p = z_{p+1} + z_p. \tag{2.150}$$

Then $A_p$ can be given by

$$A_p = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a'_p e^{-jk_x(x-x')} e^{-jk_y(y-y')} dk_x dk_y, \tag{2.151}$$

with

$$a'_p = \frac{p_x}{j2\xi_p \Delta m,VEp} \left\{ \frac{\xi_p}{k_0} \mathcal{D}^{VEp}_{11} \mathcal{W}'_1 \left[ \cos \xi_p (\Sigma z_p - (z + z')) + \cos \xi_p (\Delta z_p - |z - z'|) \right] \right.$$ 

$$+ \frac{\epsilon_p k_0}{\epsilon_{p+1} \xi_p} \mathcal{D}^{VEp}_{21} \mathcal{W}'_2 \left[ \cos \xi_p (\Sigma z_p - (z + z')) - \cos \xi_p (\Delta z_p - |z - z'|) \right]$$

$$- \frac{\epsilon_p}{\epsilon_{p+1}} \mathcal{D}^{VEp}_{11} \mathcal{W}'_{2j} \left[ \sin \xi_p (\Sigma z_p - (z + z')) + \sin \xi_p (\Delta z_p - |z - z'|) \right]$$

$$- \mathcal{D}^{VEp}_{21} \mathcal{W}'_{1j} \left[ \sin \xi_p (\Sigma z_p - (z + z')) - \sin \xi_p (\Delta z_p - |z - z'|) \right]\right\}, \tag{2.152}$$

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and the normal component of the electric field in the $p^{th}$ layer can be given by

$$E_{p,z} = -j\frac{\omega\mu_p}{k_p^2} (k_p^2 + \frac{\partial^2}{\partial z^2}) A_p.$$  

(2.153)

Applying the cylindrical coordinate transformation given in the section 2.2 to (2.151) yields

$$A_p = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^\infty a_p' e^{-j\lambda \rho \cos(\alpha - \phi)} d\lambda d\alpha.$$  

(2.154)

Noticing that the $\frac{\partial^2}{\partial z^2}$ operator in (2.153) will not change the $\alpha$-dependency of the integrand, thus integrating over $\alpha$ yields

$$A_p = \frac{1}{2\pi} \int_0^\infty a_p' J_0(\lambda \rho) \lambda d\lambda.$$  

(2.155)

In order to make it more convenient to compute the coupling between the electric field due to a vertical electric current source and another vertical electric current source, $A_p$ can be decomposed into functions of $z - z'$ and $z + z'$ by introducing the following functions:

$$V^{V_{Ep}}_{pa} = \int_0^\infty d\lambda J_0(\lambda \rho) \lambda \left\{ \frac{1}{2k_0\xi_p \Delta m, V_{Ep}} \left[ \left( \xi_p \mathcal{D}_{11}^{V_{Ep}} \mathcal{W}_1 - \frac{\epsilon_p}{\epsilon_p + 1} \xi_p \mathcal{D}_{21}^{V_{Ep}} \mathcal{W}_2 \right) \cos \xi_p(\Delta z_p - |z - z'|) \right. \right.$$  

$$
- \left. \left( \frac{\epsilon_p}{\epsilon_p + 1} \mathcal{D}_{11}^{V_{Ep}} \mathcal{W}_2 - \mathcal{D}_{21}^{V_{Ep}} \mathcal{W}_1 \right) \right\} \right.$$  

and

$$V^{V_{Ep}}_{pb} = \int_0^\infty d\lambda J_0(\lambda \rho) \lambda \left\{ \frac{1}{2k_0\xi_p \Delta m, V_{Ep}} \left[ \left( \xi_p \mathcal{D}_{11}^{V_{Ep}} \mathcal{W}_1 + \frac{\epsilon_p}{\epsilon_p + 1} \xi_p \mathcal{D}_{21}^{V_{Ep}} \mathcal{W}_2 \right) \cos \xi_p(\Sigma z_p - (z + z')) \right. \right.$$  

$$
- \left. \left( \frac{\epsilon_p}{\epsilon_p + 1} \mathcal{D}_{11}^{V_{Ep}} \mathcal{W}_2 + \mathcal{D}_{21}^{V_{Ep}} \mathcal{W}_1 \right) \right\} \right.$$  

(2.156)

(2.157)

Then $E_{p,z}$ can be rewritten in terms of these newly defined functions as:

$$E_{p,z} = -\frac{k_0\omega\mu_p}{2\pi k_p^2} \left[ k_p^2 (V_{Va}^{V_{Ep}} + V_{Vb}^{V_{Ep}}) - \frac{\partial^2 V_{Va}^{V_{Ep}}}{\partial z \partial z'} + \frac{\partial^2 V_{Vb}^{V_{Ep}}}{\partial z \partial z'} \right] \cdot p_z.$$  

(2.158)
In a case where a vertical probe extends beyond the \( p^{th} \) layer, the coupling between the electric field due to a vertical source in the \( p^{th} \) layer and a vertical current source in another layer might be needed. Such situation leads to the requirement of a knowledge of the normal component of the electric field in the layer above the \( p^{th} \) layer. It is noted that the electric fields in the layers below the \( p^{th} \) layer can be obtained by applying the reciprocity theorem to the results obtained in the previous section. From (2.137), the vector potential for the field in the \( q^{th} \) layer (\( q > p \)), \( A_q \), is given by

\[
A_q = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk_x dk_y e^{-jk_x(x-x')} e^{-jk_y(y-y')} \frac{p_z}{\Delta_{m,VEp}} \left[ C_{m'}^{q,VEp} \cos \xi_q(z - z_q) + \frac{j k_0}{\xi_q} B_{q}^{m',VEp} \sin \xi_q(z - z_q) \right], \tag{2.159}
\]

and the normal component, \( E_{q,z} \), can be given by

\[
E_{q,z} = -j \frac{\omega \mu_q}{k_q^2} (k_q^2 + \frac{\partial^2}{\partial z^2}) A_q. \tag{2.160}
\]

From (2.130) and (2.134), \( C_q^{m',VEp} \) and \( B_q^{m',VEp} \) are given by

\[
\begin{bmatrix}
C_q^{m',VEp} \\
B_q^{m',VEp}
\end{bmatrix}
= V_{q,VEp}^{V} \begin{bmatrix}
C_q^{m',VEp} \\
-\frac{\xi_q}{k_0} C_{m'}^{q,VEp}
\end{bmatrix}
= \left( \frac{D_{V_{21}}^{VEp}}{\xi_p} \sin \xi_p(z' - z_p) - \frac{D_{V_{11}}^{VEp}}{k_0} j \cos \xi_p(z' - z_p) \right) \begin{bmatrix}
V_{q,VEp}^{V_{11}} - \frac{\xi_q}{k_0} V_{q,VEp}^{V_{12}} \\
V_{q,VEp}^{V_{21}} - \frac{\xi_q}{k_0} V_{q,VEp}^{V_{22}}
\end{bmatrix}. \tag{2.161}
\]

Defining new functions \( V_{qa}^{VEp} \) and \( V_{qb}^{VEp} \) as follows:

\[
V_{qa}^{VEp} = \frac{j}{2\pi k_0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk_x dk_y e^{-jk_x(x-x')} e^{-jk_y(y-y')} \frac{1}{\Delta_{m,VEp}} \left[ C_{q}^{m',VEp} \cos \xi_q(z - z_q) + \frac{j k_0}{\xi_q} B_{q}^{m',VEp} \sin \xi_q(z - z_q) \right], \tag{2.162}
\]

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and

\[ V_{qb}^{VEP} = \frac{j}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk_x dk_y e^{-jk_x(x-x')} e^{-jk_y(y-y')} \]

\[
\frac{1}{\Delta m,VE} \left( -D_{21}^{VEP} \cos \xi_p (z' - z_p) + \frac{D_{11}^{VEP}}{k_0} j \sin \xi_p (z' - z_p) \right)
\]

\[
\left( \mathbf{V}_{11}^{q,VEP} - \frac{\xi_n}{k_0} \mathbf{V}_{12}^{q,VEP} \right) \frac{\xi_q}{k_0} \sin \xi_q (z - q) - j \left( \mathbf{V}_{21}^{q,VEP} - \frac{\xi_n}{k_0} \mathbf{V}_{22}^{q,VEP} \right) \cos \xi_q (z - q) \right],
\]

then (2.160) can be rewritten as

\[ \hat{z}E_{q,z} = -\hat{z} \frac{k_0 \omega \mu_q}{2\pi k^2 q} \left( k^2_q V_{qa}^{VEP} - \frac{\partial^2}{\partial z^2} V_{qb}^{VEP} \right) \cdot \mathbf{p}_z. \]  

(2.164)

Now, applying the cylindrical coordinate transformation given in the previous section to (2.162) and (2.163), and integrating over \( \alpha \) yields

\[ V_{qa}^{VEP} = \int_{0}^{\infty} \frac{j}{k_0 \Delta m,VE} J_0(\lambda \rho) \lambda \]

\[
\left[ C_{q}^{m',VEP} \cos \xi_q (z - q) + \frac{j k_0}{\xi_q} B_{q}^{m',VEP} \sin \xi_q (z - q) \right] d\lambda, \quad (2.165)
\]

and

\[ V_{qb}^{VEP} = -\int_{0}^{\infty} \frac{1}{\Delta m,VE} \left( D_{21}^{VEP} \cos \xi_p (z' - z_p) + \frac{D_{11}^{VEP}}{k_0} j \sin \xi_p (z' - z_p) \right) J_0(\lambda \rho) \lambda \]

\[
\left( \mathbf{V}_{11}^{q,VEP} - \frac{\xi_n}{k_0} \mathbf{V}_{12}^{q,VEP} \right) \frac{\xi_q}{k_0} j \sin \xi_q (z - q) + \left( \mathbf{V}_{21}^{q,VEP} - \frac{\xi_n}{k_0} \mathbf{V}_{22}^{q,VEP} \right) \cos \xi_q (z - q) \right] d\lambda, \quad (2.166)
\]

which can be evaluated efficiently.

Another case of interest in this case is the tangential component of the electric field at \( z = z_m, m > p \) at a different horizontal source, which is used to compute the coupling between the electric field due to a vertical source in the \( p^{th} \) layer and a horizontal source in the \( m^{th} \) layer. It can be found from (2.139) to be

\[ \hat{t}E_{m,t} = -\frac{1}{4\pi^2 \omega \epsilon_m} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk_x dk_y e^{-jk_x(x-x')} e^{-jk_y(y-y')} jk_0 B_{m}^{m',VEP} \Delta m,VEP (\hat{x}k_x + \hat{y}k_y) \cdot \mathbf{p}_z. \]  

(2.167)
\( B_{m',VEp} \) can be found from (2.134) as

\[
B_{m',VEp} = \left( \mathbf{V}_{21}^{m,VEp} - \frac{\xi_n}{k_0} \mathbf{V}_{22}^{m,VEp} \right) C_n^{m',VEp},
\]

(2.168)

where \( \mathbf{V}_{m,VEp} = \prod_{k=m}^{n-1} \mathbf{U}_k^m \). Thus,

\[
\hat{t}E_{m,t} = -\frac{1}{4\pi^2\omega\epsilon_m} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk_x dk_y e^{-jk_x(x-x')} e^{-jk_y(y-y')} \left( \mathbf{V}_{21}^{m,VEp} - \frac{\xi_n}{k_0} \mathbf{V}_{22}^{m,VEp} \right) \frac{j k_0 C_n^{m',VEp}}{\Delta_{m,VEp}} \left( \hat{x} k_x + \hat{y} k_y \right) \cdot \mathbf{p}.
\]

(2.169)

Defining the following function as follows

\[
W^{VEp} = \frac{1}{2\pi} \int_{-\alpha}^{\lambda} \frac{1}{\Delta_{m,VEp}} \left( \mathbf{V}_{21}^{m,VEp} - \frac{\xi_n}{k_0} \mathbf{V}_{22}^{m,VEp} \right)
\]

\[
\left( -\mathbf{D}_{21}^{VEp} \cos \xi_p \frac{z'-z_p}{\xi_p^2} - \mathbf{D}_{11}^{VEp} j \sin \xi_p \frac{z'-z_p}{\xi_p} \right) \frac{e^{-jk_x(x-x')} e^{-jk_y(y-y')}}{\Delta_{m,VEp}} dk_x dk_y,
\]

(2.170)

then (2.169) can be rewritten using (2.170) as

\[
\hat{t}E_{m,t} = \frac{\omega \mu_0 \epsilon_0}{2\pi \epsilon_m k_0} \left( \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} \right) \frac{z}{\Delta_{m,VEp}} \frac{\partial W^{VEp}}{\partial z'} \cdot \mathbf{p}.
\]

(2.171)

By using the cylindrical coordinate transformation and integrating over \( \alpha \), (2.170) becomes

\[
W^{VEp} = \int_{0}^{\infty} \frac{1}{\Delta_{m,VEp}} \left( \mathbf{V}_{21}^{m,VEp} - \frac{\xi_n}{k_0} \mathbf{V}_{22}^{m,VEp} \right)
\]

\[
\left( -\mathbf{D}_{21}^{VEp} \cos \xi_p \frac{z'-z_p}{\xi_p^2} - \mathbf{D}_{11}^{VEp} j \sin \xi_p \frac{z'-z_p}{\xi_p} \right) J_0(\lambda \rho) \lambda d\lambda,
\]

(2.172)

which is the form to be used in calculating the mutual coupling.

### 2.5 Field due to a Horizontal Magnetic Current Source on the Ground Plane

Using the equivalence theorem, an aperture on the ground plane can be replaced by an equivalent magnetic current source radiating in the presence of the perfect ground plane.
which also covers the aperture. Therefore, a coaxial aperture or a slot is typically modelled as a horizontal magnetic current source on the ground plane. The field due to a horizontal magnetic current source can be solved using the same approach previously described in section 2.2.

Now, let \( M \) be a magnetic current source on the ground plane given by

\[
M = m_t \delta(x - x') \delta(y - y') \delta(z - z') \triangleq M_s \delta(z - z'),
\]

(2.173)

where \( m_t = \hat{x} m_x + \hat{y} m_y \), and \( z' = 0 \). In the spectral domain, it becomes

\[
M_s = \frac{1}{4\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} m_t e^{-jk_x(x-x')} e^{-jk_y(y-y')} dk_x dk_y.
\]

(2.174)

Representing the vector potential in each layer in the same way as done in section 2.2, then applying boundary condition \( \hat{z} \times E_1 = -M_s \) at \( z = z_1 = 0 \) yields

\[
jk_y C_1^e - \frac{k_x}{\omega \epsilon_1} jk_0 B_1^m = -m_y e^{j\psi},
\]

(2.175)

\[
-jk_x C_1^e - \frac{k_y}{\omega \epsilon_1} jk_0 B_1^m = m_x e^{j\psi},
\]

(2.176)

where \( \psi = k_x x' + k_y y' \). From (2.175) and (2.176), one obtains

\[
B_1^{m,HM} = -\frac{\omega \epsilon_1}{k_0} \frac{je^{j\psi}}{k_x^2 + k_y^2} (k_x m_y - k_y m_x),
\]

(2.177)

and

\[
C_1^{e,HM} = \frac{je^{j\psi}}{k_x^2 + k_y^2} (k_x m_x + k_y m_y).
\]

(2.178)

The \( HM \) in the superscript here denotes the parameters for the field due to the horizontal magnetic current source. Now, applying the boundary conditions at other layer boundaries
and radiation condition in the half space region yields

\[
\begin{bmatrix}
C_{m,HM}^m \\
B_{1m,HM}^m
\end{bmatrix}
= \mathcal{U}_m \mathcal{U}_2 \cdots \mathcal{U}_{n-1} \begin{bmatrix}
C_{m,HM}^n \\
B_{1n,HM}^n
\end{bmatrix}
\]
\[
= \mathcal{U}_m^{m,HM} \begin{bmatrix}
C_{m,HM}^n & -\xi_n C_{m,HM}^n \\
-\frac{\xi_n}{k_0} & -\frac{\xi_n}{k_0}
\end{bmatrix}
\]
\[
= C_n^{m,HM} \begin{bmatrix}
\mathcal{U}_{11}^{m,HM} & -\frac{\xi_n}{k_0} \mathcal{U}_{12}^{m,HM} \\
\mathcal{U}_{21}^{m,HM} & -\frac{\xi_n}{k_0} \mathcal{U}_{22}^{m,HM}
\end{bmatrix},
\tag{2.179}
\]

and

\[
\begin{bmatrix}
C_{e,HM}^m \\
B_{1e,HM}^m
\end{bmatrix}
= \mathcal{U}_e \mathcal{U}_2 \cdots \mathcal{U}_{n-1} \begin{bmatrix}
C_{e,HM}^n \\
B_{1n,HM}^n
\end{bmatrix}
\]
\[
= \mathcal{U}_e^{e,HM} \begin{bmatrix}
C_{e,HM}^n & -\xi_n C_{e,HM}^n \\
-\frac{\xi_n}{k_0} & -\frac{\xi_n}{k_0}
\end{bmatrix}
\]
\[
= C_n^{e,HM} \begin{bmatrix}
\mathcal{U}_{11}^{e,HM} & -\frac{\xi_n}{k_0} \mathcal{U}_{12}^{e,HM} \\
\mathcal{U}_{21}^{e,HM} & -\frac{\xi_n}{k_0} \mathcal{U}_{22}^{e,HM}
\end{bmatrix}.
\tag{2.180}
\]

From (2.177) and (2.179), one obtains

\[
\begin{bmatrix}
C_{m,HM}^n \\
C_{1m,HM}
\end{bmatrix}
= \begin{bmatrix}
C_{n}^{m',HM} \\
C_{1}^{m',HM}
\end{bmatrix} \frac{B_{1m,HM}^{m',HM}}{\Delta_{m,HM}}
\]
\[
= \begin{bmatrix}
C_{n}^{m',HM} \\
C_{1}^{m',HM}
\end{bmatrix} \frac{\omega \epsilon_1}{k_0} \frac{j e^{j\psi}}{\Delta_{m,HM}} \frac{k_y m_x - k_x m_y}{k_x^2 + k_y^2},
\tag{2.181}
\]

where

\[
C_{n}^{m',HM} = 1,
\tag{2.182}
\]
\[
C_{1}^{m',HM} = \mathcal{U}_{11}^{m,HM} - \frac{\xi_n}{k_0} \mathcal{U}_{12}^{m,HM},
\tag{2.183}
\]
\[
\Delta_{m,HM} = \mathcal{U}_{21}^{m,HM} - \frac{\xi_n}{k_0} \mathcal{U}_{22}^{m,HM}.
\tag{2.184}
\]

Likewise, from (2.178) and (2.180), one obtains

\[
\begin{bmatrix}
C_{e,HM}^n \\
B_{1e,HM}^n
\end{bmatrix}
= \begin{bmatrix}
C_{n}^{e',HM} \\
B_{1}^{e',HM}
\end{bmatrix} \frac{C_{1}^{e,HM}}{\Delta_{e,HM}}
\]
\[
= \begin{bmatrix}
C_{n}^{e',HM} \\
B_{1}^{e',HM}
\end{bmatrix} \frac{j e^{j\psi}}{\Delta_{e,HM}} \frac{k_x m_x + k_y m_y}{k_x^2 + k_y^2},
\tag{2.185}
\]

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where

\[ C_n^{e',HM} = 1, \]  
\[ B_1^{e',HM} = U_{21}^{e,HM} - \frac{\xi_n}{k_0} U_{22}^{e,HM}, \]  
\[ \Delta^{e,HM} = U_{11}^{e,HM} - \frac{\xi_n}{k_0} U_{12}^{e,HM}. \]

The other coefficients can be found from

\[
\begin{bmatrix}
  C_{i}^{m',HM} \\
  B_{i}^{m',HM}
\end{bmatrix} = \prod_{k=i}^{n-1} U_k^{m} \begin{bmatrix}
  C_{n}^{m',HM} \\
  \frac{\xi_n}{k_0} C_{n}^{m',HM}
\end{bmatrix},
\]

\[
\begin{bmatrix}
  C_{i}^{e',HM} \\
  B_{i}^{e',HM}
\end{bmatrix} = \prod_{k=i}^{n-1} U_k^{e} \begin{bmatrix}
  C_{n}^{e',HM} \\
  \frac{\xi_n}{k_0} C_{n}^{e',HM}
\end{bmatrix},
\]

Let \( B_{1}^{m',HM} = \Delta^{m,HM} \) and \( C_{1}^{e',HM} = \Delta^{e,HM} \), then \( a_i, f_i \) in spectral domain can be written as

\[
a_i = -\frac{\omega e_1}{k_0} j e^{j(k_x x' + k_y y')} \frac{k_x m_y - k_y m_x}{k_x^2 + k_y^2} \frac{1}{\Delta^{m,HM}} \cos \xi_i(z - z_i) + j \frac{k_0}{\xi_i} B_{i}^{m',HM} \sin \xi_i(z - z_i),
\]

and

\[
f_i = j e^{j(k_x x' + k_y y')} \frac{k_x m_y + k_y m_x}{k_x^2 + k_y^2} \frac{1}{\Delta^{e,HM}} \{ C_{i}^{e',HM} \cos \xi_i(z - z_i) + j \frac{k_0}{\xi_i} B_{i}^{e',HM} \sin \xi_i(z - z_i) \},
\]

respectively.

Now, the dyadic Green’s function relates to the electric field and the source by

\[
E_i = \overline{G_i} \cdot m_t.
\]
thus, the multilayered media Green’s function for the electric field due to a horizontal magnetic current source on the ground plane in the $i^{th}$ layer can be found by inspection as

$$
\mathbf{G}_i^{HM}(\mathbf{r} | \mathbf{r}') = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \overline{Y}_i^{HM}(k_x, k_y) e^{-jk_x(x-x')} e^{-jk_y(y-y')} dk_x dk_y,
$$

(2.194)

where

$$
\overline{Y}_i^{HM}(k_x, k_y) = \chi_m^{t,i} \overline{\chi}_m^{i} + \chi_m^{z,i} \overline{\chi}_z^{i} + \chi_e^{t,i} \overline{\chi}_e^{i},
$$

(2.195)

$$
\overline{Y}_i^{m,HM}(\xi_i) = \frac{\epsilon_1}{jk_0 \epsilon_m \Delta m,HM} \left[ -C_i^{m',HM} \xi_i \sin \xi_i(z - z_i) + jk_0 B_i^{m',HM} \cos \xi_i(z - z_i) \right],
$$

(2.196)

$$
\overline{Y}_i^{m,HM}(\xi_i) = \frac{1}{k_0 \epsilon_i \Delta m,HM} \left[ C_i^{m,HM} \cos \xi_i(z - z_i) + jB_i^{m,HM} \frac{k_0}{\xi_i} \sin \xi_i(z - z_i) \right],
$$

(2.197)

$$
\overline{Y}_i^{e,HM}(\xi_i) = \frac{1}{\Delta e,HE} \left[ C_i^{e',HM} \cos \xi_i(z - z_i) + jB_i^{e',HM} \frac{k_0}{\xi_i} \sin \xi_i(z - z_i) \right],
$$

(2.198)

$$
\overline{\chi}_m^{i,HM}(k_x, k_y) = \frac{1}{k_x^2 + k_y^2} \left[ \hat{x} \hat{x} k_x k_y + \hat{y} \hat{y} k_x^2 - \hat{y} \hat{y} k_y^2 \right],
$$

(2.199)

$$
\overline{\chi}_z^{i,HM}(k_x, k_y) = \hat{z} \hat{x} k_x - \hat{z} \hat{y} k_y,
$$

(2.200)

$$
\overline{\chi}_e^{i,HM}(k_x, k_y) = \frac{1}{k_x^2 + k_y^2} \left[ -\hat{x} \hat{x} k_x k_y - \hat{y} \hat{y} k_x^2 + \hat{y} \hat{y} k_y^2 \right].
$$

(2.201)

When a magnetic current source is used as an excitation, an element of the excitation vector is given by the reaction between the electric field due to this current and a test current, which can be either a horizontal electric current or a vertical electric current. Therefore, of interest are the tangential components of the electric field at $z = z_m$, where a horizontal current source is located, and the normal component component of the electric field in the layer in which a vertical current is embedded.
After some manipulations, the tangential component of the electric field at \( z = z_m \) can be given by

\[
\hat{E}_{m,t} = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk_x dk_y e^{-j\kappa_s(x-x')} e^{-j\kappa_y(y-y')} \left[ (\hat{y}\hat{x} - \hat{x}\hat{y}) \frac{C_{m}^{e,HM}}{\Delta e_{HM}} \right. \\
- \left. (\hat{x}\hat{x}k_x k_y - \hat{x}\hat{y}k_y^2 + \hat{y}\hat{x}k_x^2 - \hat{y}\hat{y}k_x k_y) \frac{1}{k_x^2 + k_y^2} \left( \frac{C_{m}^{e,HM}}{\Delta e_{HM}} - \frac{\epsilon_1 B_m^{m',HM}}{\epsilon_m \Delta m_{HM}} \right) \right] \cdot \mathbf{m}_t, \tag{2.202}
\]

which can be rewritten as

\[
\hat{E}_{m,t} = \frac{1}{2\pi} \left[ (\hat{y}\hat{x} - \hat{x}\hat{y}) k_0^2 U^{HM} - \left( \hat{x}\hat{x} \frac{\partial}{\partial x} + \hat{x}\hat{y} \frac{\partial}{\partial y} \right) k_0 W_y^{HM} + \right. \\
\left. \left( \hat{x}\hat{y} \frac{\partial}{\partial x} + \hat{y}\hat{x} \frac{\partial}{\partial y} \right) k_0 W_x^{HM} \right] \cdot \mathbf{m}_t, \tag{2.203}
\]

where

\[
U^{HM} = \frac{1}{2\pi k_0^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{C_{m}^{e,HM}}{\Delta e_{HM}} e^{-j\kappa_s(x-x')} e^{-j\kappa_y(y-y')} dk_x dk_y, \tag{2.204}
\]

\[
\begin{align*}
\left\{ \frac{W_x^{HM}}{W_y^{HM}} \right\} &= \frac{1}{2\pi k_0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ k_x \right\} \left( \frac{C_{m}^{e,HM}}{\Delta e_{HM}} - \frac{\epsilon_1 B_m^{m',HM}}{\epsilon_m \Delta m_{HM}} \right) \\
&\quad \frac{j}{k_x^2 + k_y^2} e^{-j\kappa_s(x-x')} e^{-j\kappa_y(y-y')} dk_x dk_y. \tag{2.205}
\end{align*}
\]

Applying the cylindrical coordinate transformation to (2.204), and (2.205) yields

\[
U^{HM} = \frac{1}{2\pi k_0^2} \int_{0}^{\infty} \int_{0}^{2\pi} \frac{C_{m}^{e,HM}}{\Delta e_{HM}} e^{-j\lambda \rho \cos(\alpha - \phi)} \lambda d\alpha d\lambda, \tag{2.206}
\]

\[
\begin{align*}
\left\{ \frac{W_x^{HM}}{W_y^{HM}} \right\} &= \frac{1}{2\pi k_0} \int_{0}^{\infty} \int_{0}^{2\pi} \left\{ \lambda \cos \alpha \right\} \left( \frac{C_{m}^{e,HM}}{\Delta e_{HM}} - \frac{\epsilon_1 B_m^{m',HM}}{\epsilon_m \Delta m_{HM}} \right) \\
&\quad \frac{j}{\lambda^2} e^{-j\lambda \rho \cos(\alpha - \phi)} \lambda d\alpha d\lambda. \tag{2.207}
\end{align*}
\]

Integrating the right-hand side of (2.206) over \( \alpha \) yields

\[
U^{HM} = \frac{1}{k_0^2} \int_{0}^{\infty} \frac{C_{m}^{e,HM}}{\Delta e_{HM}} J_0(\lambda \rho) \lambda d\lambda. \tag{2.208}
\]
Now, using the following identities
\[ \frac{1}{2\pi} \int_{0}^{2\pi} \begin{bmatrix} \cos \alpha \\
\sin \alpha \end{bmatrix} e^{-j\lambda \rho \cos(\alpha - \phi)} d\alpha = -j \begin{bmatrix} \cos \phi \\
\sin \phi \end{bmatrix} J_1(\lambda \rho), \]
(2.209)
where \( J_1(x) \) is the Bessel function of the first order, and integrating the right-hand side of (2.207) over \( \alpha \), one obtains
\[ \begin{bmatrix} W_{x}^{HM} \\
W_{y}^{HM} \end{bmatrix} = \frac{1}{k_0} \int_{0}^{\infty} \begin{bmatrix} \cos \phi \\
\sin \phi \end{bmatrix} \left( \frac{C_{m}^{e,HM}}{\Delta \epsilon,HM} - \frac{\epsilon_1 B_{m}^{m',HM}}{\epsilon_m \Delta \mu,HM} \right) J_1(\lambda \rho) d\lambda, \]
(2.210)
which can be rewritten as
\[ \begin{bmatrix} W_{x}^{HM} \\
W_{y}^{HM} \end{bmatrix} = \begin{bmatrix} \cos \phi \\
\sin \phi \end{bmatrix} W^{HM}, \]
(2.211)
where
\[ W^{HM} = \frac{1}{k_0} \int_{0}^{\infty} \left( \frac{C_{m}^{e,HM}}{\Delta \epsilon,HM} - \frac{\epsilon_1 B_{m}^{m',HM}}{\epsilon_m \Delta \mu,HM} \right) J_1(\lambda \rho) d\lambda. \]
(2.212)
Likewise, using (2.194), the normal component of the electric field in the \( q^{th} \) layer can be given by
\[ \hat{z}E_{q,z} = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \epsilon_1 e^{-\frac{k_0}{k_0 e_q}} \left( \frac{1}{\Delta \mu,HM} \left[ C_{q}^{m',HM} \cos \xi_q (z - z_q) + jk_0 B_{q}^{m',HM} \sin \xi_q (z - z_q) \right] \right) \cdot \mathbf{m}_t, \]
(2.213)
which can be written as
\[ \hat{z}E_{q,z} = \frac{1}{2\pi} \epsilon_1 k_0^2 \left( \hat{z}x V_{q,y}^{HM} - \hat{z}y V_{q,x}^{HM} \right) \cdot \mathbf{m}_t, \]
(2.214)
with
\[ \begin{bmatrix} V_{q,x}^{HM} \\
V_{q,y}^{HM} \end{bmatrix} = \frac{1}{2\pi k_0^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \epsilon_1 e^{-\frac{k_0}{k_0 e_q}} \left( \frac{k_x}{k_y} \right) \left[ C_{q}^{m',HM} \cos \xi_q (z - z_q) + jk_0 B_{q}^{m',HM} \sin \xi_q (z - z_q) \right]. \]
(2.215)
Applying the cylindrical coordinate transformation to the above equation yields

\[
\begin{align*}
\left\{ V_{q,x}^{HM}, V_{q,y}^{HM} \right\} &= \frac{1}{2\pi k_0^3} \int_0^\infty \int_0^{2\pi} \left[ C_{q}^{m',HM} \cos \xi_q (z - z_q) + \frac{j k_0}{\xi_q} B_{q}^{m',HM} \sin \xi_q (z - z_q) \right] \\
&\quad \left\{ \frac{\lambda \cos \alpha}{\lambda \sin \alpha} \right\} \frac{1}{\Delta^{m,HM}} e^{-j \lambda \rho \cos (\alpha - \phi)} \lambda d\alpha d\lambda. 
\end{align*}
\]

(2.216)

By integrating over \(\alpha\) using (2.209), one obtains

\[
\begin{align*}
\left\{ V_{q,x}^{HM}, V_{q,y}^{HM} \right\} &= \frac{1}{k_0^3} \int_0^\infty \left\{ \frac{\cos \phi}{\sin \phi} \right\} \frac{-j}{\Delta^{m,HM}} J_1 (\lambda \rho) \lambda^2 \\
&\quad \left[ C_{q}^{m',HM} \cos \xi_q (z - z_q) + \frac{j k_0}{\xi_q} B_{q}^{m',HM} \sin \xi_q (z - z_q) \right] d\lambda \\
&= \left\{ \frac{\cos \phi}{\sin \phi} \right\} V_{q}^{HM}, \quad (2.217)
\end{align*}
\]

with

\[
V_{q}^{HM} = -\frac{1}{k_0^3} \int_0^\infty \frac{j}{\Delta^{m,HM}} J_1 (\lambda \rho) \lambda^2 \\
&\quad \left[ C_{q}^{m',HM} \cos \xi_q (z - z_q) + \frac{j k_0}{\xi_q} B_{q}^{m',HM} \sin \xi_q (z - z_q) \right] d\lambda. \quad (2.218)
\]

(2.204), (2.212) and (2.218) are the main results used to calculate the coupling between the electric field due to a horizontal magnetic current source on the ground plane and an electric current source.

In electromagnetic problems where slot antenna elements are of interest, one can apply the equivalence theorem and replace the slots in the ground plane with equivalent magnetic currents on the ground plane which now close the slots as well. A magnetic field integral equation, where the magnetic currents become unknown, can be formulated for this situation. In this case, the coupling between slots is given by the reaction of the magnetic field due to a magnetic current and a test magnetic current, which requires the computation of the magnetic field instead of the electric field. Since the source is on the ground plane, it is
clear that only the magnetic field in the bottommost layer, denoted here by $H_1$, is needed.

$H_1$ can be found from (2.4) as

$$H_1 = \nabla \times A_1 - j\omega \varepsilon_1 F_1 + \frac{1}{j\omega \mu_1} \nabla \nabla \cdot F_1.$$  

Using (2.191) and (2.192) yields

$$\nabla \times A_1 = \frac{\varepsilon_1}{4\pi^2 k_0} \int_{-\infty}^{\infty} dk_x dk_y e^{-jk_x (x-x')} e^{-jk_y (y-y')} \left( \frac{B_1^{m',HM} \cos \xi_1 z + k_0 \frac{B_1^{m',HM}}{\xi_1} \sin \xi_1 z}{k_x^2 + k_y^2} \right)$$

$$\frac{1}{\Delta m, HM} \left( \hat{x} \hat{x} k_x^2 + \hat{y} \hat{y} k_y^2 \right) \cdot m_t, \quad (2.219)$$

and

$$\frac{1}{j \omega \mu_1} \nabla \nabla \cdot F_1 = \frac{k_0}{4\pi^2 \omega \mu_1} \int_{-\infty}^{\infty} dk_x dk_y e^{-jk_x (x-x')} e^{-jk_y (y-y')} \left( \frac{B_1^{e',HM} \cos \xi_1 z + jC_1^{e',HM} \sin \xi_1 z}{\xi_1} \right)$$

$$\frac{1}{\Delta e, HM} \left( \hat{x} \hat{x} k_x^2 + \hat{y} \hat{y} k_y^2 \right) \cdot m_t. \quad (2.220)$$

Using the above results, the tangential component of the magnetic field on the ground plane,

i.e., at $z = 0$, can be given by

$$\hat{t}H_{1,t} = \frac{\varepsilon_0}{\mu_0 4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk_x dk_y e^{-jk_x (x-x')} e^{-jk_y (y-y')}$$

$$\left( \frac{B_1^{e',HM}}{\Delta e, HM} - \frac{\varepsilon_1 C_1^{m',HM}}{\epsilon_0 \Delta m, HM} \right) \cdot m_t. \quad (2.221)$$

Now, introducing the following functions

$$S^{HM} = \frac{1}{2\pi k_0^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk_x dk_y e^{-jk_x (x-x')} e^{-jk_y (y-y')} \frac{\varepsilon_1 C_1^{m',HM}}{\epsilon_0 \Delta m, HM}, \quad (2.222)$$

and

$$T^{HM} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk_x dk_y e^{-jk_x (x-x')} e^{-jk_y (y-y')} \frac{1}{k_x^2 + k_y^2} \left( \frac{B_1^{e',HM}}{\Delta e, HM} - \frac{\varepsilon_1 C_1^{m',HM}}{\epsilon_0 \Delta m, HM} \right), \quad (2.223)$$
then (2.221) becomes

\[
\hat{H}_{1,t} = \sqrt{\frac{\varepsilon_0}{\mu_0}} \frac{1}{2\pi} \left[ (\hat{x}\hat{x} + \hat{y}\hat{y}) k_0^2 S^{HM} + \left( \hat{x}\hat{x} \frac{\partial^2}{\partial x^2} + \hat{x}\hat{y} \frac{\partial^2}{\partial x \partial y} + \hat{y}\hat{x} \frac{\partial^2}{\partial y \partial x} + \hat{y}\hat{y} \frac{\partial^2}{\partial y^2} \right) T^{HM} \right] \cdot m_t
\]

\[
= \sqrt{\frac{\varepsilon_0}{\mu_0}} \frac{1}{2\pi} \left[ (\hat{x}\hat{x} + \hat{y}\hat{y}) k_0^2 S^{HM} + \frac{1}{k_0^2} \nabla_t \nabla'_t T^{HM} \right] \cdot m_t. \tag{2.224}
\]

Now, using the cylindrical coordinate transformation and integrating over \( \alpha \) yields

\[
S^{HM} = \frac{1}{k_0^2} \int_0^\infty \frac{\varepsilon_1}{\varepsilon_0} \frac{C_{1,m'.HM}^m}{\Delta_{m'.HM}} J_0(\lambda \rho) \lambda d\lambda \tag{2.225}
\]

and

\[
T^{HM} = \int_0^\infty \left( \frac{B_{1,m'.HM}^{e.HM}}{\Delta_{e.HM}} - \frac{\varepsilon_1}{\varepsilon_0} \frac{C_{1,m'.HM}^m}{\Delta_{m'.HM}} \right) \frac{J_0(\lambda \rho)}{\lambda} d\lambda, \tag{2.226}
\]

which will be the representations used in the computation of coupling or the MoM operator matrix elements.

Table 2.1 summarizes all functions used to represent the multilayered media Green’s functions developed in this chapter. Note that a horizontal electric current source is located at \( z = z_m \) i.e., the \( m^{th} \) layer is the source layer and the tangential electric field implies the field at \( z = z_m \).
Table 2.1: List of functions associated with the multilayered media Green’s functions

<table>
<thead>
<tr>
<th>Source Type</th>
<th>Field Component</th>
<th>Functions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Horizontal electric current</td>
<td>Tangential electric</td>
<td>$U_{HE}, W_{HE}$</td>
</tr>
</tbody>
</table>
| Vertical electric current in the first layer | Tangential electric  
Normal electric in first layer  
Normal electric in $q^{th}$ layer | $W_{VE1}$, $V_{VE1}$, $V_{VE1}^{qa}$, $V_{VE1}^{qb}$ |
| Vertical electric current in the $p^{th}$ layer | Tangential electric  
Normal electric in $p^{th}$ layer  
Normal electric in $q^{th}$ layer | $W_{VEp}$, $V_{VEp}$, $V_{VEp}^{qa}$, $V_{VEp}^{qb}$ |
| Horizontal magnetic current on the ground plane | Tangential electric  
Normal electric in $q^{th}$ layer  
Tangential magnetic on the ground plane | $U_{HM}, W_{HM}^{x}, W_{HM}^{y}$, $V_{HM}^{qa,x}, V_{HM}^{qa,y}$, $S_{HM}^{q}, T_{HM}^{q}$ |
CHAPTER 3

EVALUATION OF MULTILAYERED MEDIA DYADIC GREEN’S FUNCTION

As can be seen from the development in the previous chapter, the multilayered media dyadic Green’s function can be expressed in terms of infinite spectral integrals. The complexities in evaluating these integrals lie not only in the infinite integration range with slowly-decaying integrands, but also in the oscillatory nature of the integrands. To overcome these difficulties, the discrete complex image method (DCIM) [38, 57, 58] has been developed by others to transform the spectral-domain integral representation into a closed-form spatial-domain integral representation by first approximating the integrand in terms of a summation of complex exponentials and then applying the Sommerfeld identity. This method is found to be reasonably accurate, but the computational cost required to extract the complex images can be expensive and it might need a large number of terms in order to obtain correct results for a large separation between source and observation points.

Alternatively, it is found in [59–61] that single-layer or double-layer media dyadic Green’s functions can be evaluated relatively easily by asymptotic closed-form representations, which can give excellent results for lateral source and observation point separations larger than one free-space wavelength (denoted by $\lambda_0$) with only few terms. The large parameter in this asymptotic development is $k_0|\rho - \rho'|$ where $\rho$ and $\rho'$ are source and observer
distances measured laterally (or the radial distances in cylindrical coordinates) and $k_0$ is the free space wave number. Thus, it becomes much more efficient when computing the coupling between elements with moderate to large separations. However, this representation will fail in the extreme near-zone regions. Therefore, a numerical integration technique must be applied to evaluate the spectral integral for very small separations. Since the oscillatory behavior of the integrand is not so severe when the separation is small, the integral can be evaluated efficiently by extracting proper singularities and by utilizing the proper large argument asymptotic approximations.

In this study, a numerical integration approach, which accelerates the calculation of the integration, is developed to compute the multilayered media dyadic Green’s function for small separations, while an asymptotic approximation will be developed to obtain a closed-form Green’s function for moderate to large separations even in the more complex case of multilayered media. This chapter consists of three major parts. The first part describes the numerical integration method used in this work for evaluating the multilayered media dyadic Green’s function, which can be applied to the Green’s function pertaining to both electric current sources and magnetic current sources developed in the previous chapter. The second part discusses the asymptotic approximation method for the multilayered media. Some numerical results will be presented in the third part to conclude this chapter.

3.1 The Conventional Approach for the Numerical Integration Method and its Limitations

The main objective here is to numerically evaluate the spectral integrals associated with the terms contained in the multilayered media dyadic Green’s function, e.g., $U^{HE}$, $W^{HE}$, etc, in an efficient manner. Notice that these integrals can be written in the following
general form:
\[ F = \int_0^\infty f(\xi_1, \cdots, \xi_n) g(\lambda) J_i(\lambda \rho) d\lambda, \]  
(3.1)

where \( J_i(\lambda \rho) \), \( i = 0 \) or 1 denotes the Bessel function of zeroth or first order, \( g(\lambda) \) is given by
\[
g(\lambda) = \begin{cases} 
\frac{1}{\lambda} & \text{for } W^{HE}, T^{HM} \\
1 & \text{for } W^{HM} \\
\lambda^2 & \text{for } V^{HM} \\
\lambda & \text{otherwise}
\end{cases},
\]
(3.2)
and \( f(\xi_1, \cdots, \xi_n) \) represents the rest of the integrand with \( \xi_i = \sqrt{k_i^2 - \lambda^2}, i = 1, \cdots, n \).

For large \( \lambda, \xi_i, \forall i \) can be approximated by \( -j\lambda \) and thus \( f(\xi_1, \cdots, \xi_n) \) can be approximated by an appropriate function of only \( \lambda \). Now, let \( f^\infty(\lambda) \) be the large argument approximation of \( f(\xi_1, \cdots, \xi_n) \), i.e., \( f^\infty(\lambda) = \lim_{\lambda \to \infty} f(\xi_1, \cdots, \xi_n) \), then for the range beyond a significantly large \( \lambda \), denoted here by \( \lambda_m \), where the difference between \( f(\xi_1, \cdots, \xi_n) \) and \( f^\infty(\lambda) \) is negligible, the integrand can be replaced by \( f^\infty(\lambda) g(\lambda) J_i(\lambda) \). Hence, the right-hand side of (3.1) can be rewritten as
\[
\int_0^\infty f(\xi_1, \cdots, \xi_n) g(\lambda) J_i(\lambda \rho) d\lambda \\
\approx \int_0^{\lambda_m} f(\xi_1, \cdots, \xi_n) g(\lambda) J_i(\lambda \rho) d\lambda + \int_{\lambda_m}^\infty f^\infty(\lambda) g(\lambda) J_i(\lambda \rho) d\lambda,
\]
(3.3)
whose second term in the right-hand side can in turn be written as
\[
\int_{\lambda_m}^\infty f^\infty(\lambda) g(\lambda) J_i(\lambda \rho) d\lambda = \int_0^\infty f^\infty(\lambda) g(\lambda) J_i(\lambda \rho) d\lambda - \int_0^{\lambda_m} f^\infty(\lambda) g(\lambda) J_i(\lambda \rho) d\lambda.
\]
(3.4)
Thus (3.1) can be rewritten as
\[
F \approx \int_0^{\lambda_m} [f(\xi_1, \cdots, \xi_n) - f^\infty(\lambda)] g(\lambda) J_i(\lambda \rho) d\lambda + \int_0^\infty f^\infty(\lambda) g(\lambda) J_i(\lambda \rho) d\lambda.
\]
(3.5)
In general, \( f^\infty(\lambda) \) can be chosen such that a closed-form solution for the integral exists for the second term in the right-hand side of the above equation, thus only the first term is left to be evaluated numerically.
Due to the multilayered structure, the integrand of the first term will contain surface wave and leaky wave poles, i.e., roots of \( \Delta^{m,e} = 0 \). Thus, the integrand will be a fast-varying function of \( \lambda \) when \( \lambda \) approaches the poles and is difficult to evaluate numerically. To mitigate this problem, the integral can be divided into two parts as follows

\[
\int_{0}^{\lambda_m} \{ \bullet \} d\lambda = \int_{0}^{\lambda_l} \{ \bullet \} d\lambda + \int_{\lambda_l}^{\lambda_m} \{ \bullet \} d\lambda, \tag{3.6}
\]

where \( \lambda_l \) is chosen such that the effects of poles in the integration range from \( \lambda_l \) to \( \lambda_m \) is insignificant, and the integral can thus be integrated numerically without any difficulty. The integration path in \( \lambda \)-plane is shown in figure 3.1(a), where \( C_1 \) and \( C_2 \) denote the integration paths for the first and second terms in the right hand side of (3.6), respectively. Now, to evaluate the integral from 0 to \( \lambda_l \), first use the change of variable \( \lambda = k_0 \sin \gamma \) to bring up all poles into one complex \( \gamma \)-plane, as shown in figure 3.1(b), and rewrite it as

\[
A = \int_{0}^{\gamma_l} \{ \bullet \} d\gamma = \int_{0}^{\gamma_l} \frac{N(\gamma)}{D(\gamma)} d\gamma, \tag{3.7}
\]

where

\[
\gamma_l = \sin^{-1} \frac{\lambda_l}{k_0} = \frac{\pi}{2} + j\delta_l. \tag{3.8}
\]

Let \( \gamma_p = \alpha_p + j\beta_p \) denote a pole location in \( \gamma \)-plane, then (3.7) can be rewritten as

\[
A = \int_{0}^{\gamma_l} \left[ \frac{N(\gamma)}{D(\gamma)} - \sum_{p=1}^{P} \frac{N(\gamma_p)}{D'(\gamma_p)(\gamma - \gamma_p)} \right] d\gamma + \sum_{p=1}^{P} \frac{N(\gamma_p)}{D'(\gamma_p)} \int_{0}^{\gamma_l} \frac{d\gamma}{\gamma - \gamma_p}, \tag{3.9}
\]

where \( P \) denotes the total number of surface and leaky wave poles. In the first integration of (3.9), the fast-varying part, i.e., the part close to the poles, has been subtracted from the integrand, and thus it can be evaluated numerically. The second term can be evaluated
analytically by following the approach given in [62] to be

\[ \int_0^{\gamma_l} \frac{d\gamma}{\gamma - \gamma_p} = \frac{1}{2} \ln \left( \frac{\alpha_p - \pi/2}{\alpha_p^2 + \beta_p^2} \right) + \frac{(\beta_p - \delta_l)^2}{\alpha_p^2 + \beta_p^2} - j \left\{ \pi \right\} \pm j \left[ \tan^{-1} \frac{\alpha_p - \pi/2}{\delta_l - \beta_p} + \tan^{-1} \frac{\alpha_p}{\beta_p} \right], \]

where the top result is for surface wave poles and the bottom one is for leaky wave poles.

The details of this derivation are given in A.5. Although the numerical integration approach discussed here seems complicated, it is found to be very efficient in calculating the multilayered media dyadic Green’s function for small \( \rho \). It is also noted that if the poles and residues are not correct, the singularities at the poles will not be subtracted properly, and thus in that case the numerical integration along \( C_1 \) path (0 \( \leq \lambda \leq \lambda_l \)) will not converge. Therefore, this above procedure has an advantage of providing a self-checking mechanism as well.

**Limitations:**

The above numerical approach is useful only for small lateral separations of the source and field points (i.e., for \( k_0 |\vec{\rho} - \vec{\rho}'| \) small). For moderate to large separations the integrand to be evaluated numerically becomes highly oscillatory and hence very cumbersome to treat numerically. It is for the latter reason that a useful, closed form asymptotic approximation to the Green’s function spectral integral which is valid for \( k_0 |\vec{\rho} - \vec{\rho}'| > 10 \) is developed next.
Figure 3.1: Integration path used in the numerical integration
3.2 Pole Extraction

As described in the previous section, the integration path used in the numerical integration can be extremely close to the poles, thus the complete knowledge of the pole locations and their corresponding residues is crucial. Since either incorrect location or residue leads to the difficulty in the integration along path $C_1$, exact locations and accurate residues are evidently critical in this approach.

It is noted that the discrete complex image method which extracts the surface wave contributions also requires knowledge of poles and residues, prompting research regarding pole extraction. Ling et.al.[63] proposed extracting poles by recursively performing contour integrals, however that approach may require a large number of sampling points when the contour becomes very close to a pole, or may exhibit computational difficulty when two poles are close to each other. Teo et.al.[64] used the generalized pencil of function (GPOF) method to extract poles from the contour integral which encloses all poles. Although it is more general, the computational cost of GPOF can be expensive and it may not give exact pole locations due to finite precision used in the computation.

In this work, a much simpler extraction method is developed. It is first noticed that by using the change of variable $\lambda = k_0 \sin \gamma$, the branch cut can be eliminated and both surface wave and leaky wave poles will be located on their corresponding proper and improper sheets, which now map onto the same $\gamma$ plane as $\text{Im} \gamma > 0$ and $\text{Im} \gamma < 0$ regions, respectively. Thus, the search regions for each pole can be specified and the modified Newton-Raphson algorithm becomes applicable. The approach can be summarized as follows:
1. Select a point in the search region to be used as the initial guess. The search regions for both surface wave poles and leaky waves are shown in figure 3.2.

2. Apply the modified Newton-Raphson algorithm to search for a pole.

3. Repeat until the search region is exhausted.

It can be found that the results obtained by this method agree with those reported in [63] and [64].

![Figure 3.2: Search regions for surface wave and leaky wave poles in \( \gamma \)-plane](image-url)
3.3 Evaluation of Multilayered Media Dyadic Green’s Function by the Asymptotic Closed-Form Approximation Method

In this section, an efficient asymptotic approximation method used to calculate the multilayered media dyadic Green’s function will be developed. For large value of \( k_0 \rho (k_0 \rho > 10) \), the approximation technique known as the Method of Steepest Descent Path (SDP) can be applied to obtain the asymptotic expression for the spectral-domain integral representation of the multilayered media dyadic Green’s function. This approximation is found to be very accurate for the range \( \rho \geq 1.5 \lambda_0 \), where \( \lambda_0 \) denotes the free-space wavelength, with much smaller computational cost compared to the discrete complex image method or a direct numerical integration method for evaluating the multilayered Green’s function.

To apply this asymptotic technique, it is preferable to have the integration range from \(-\infty\) to \(\infty\). This can be achieved by using the following identities

\[
J_i(\lambda \rho) = \frac{1}{2} [H_i^{(1)}(\lambda \rho) + H_i^{(2)}(\lambda \rho)],
\]

(3.11)

where \( H_i^{(1)}, H_i^{(2)} \) denote the Hankel function of the \( i^{th} \) order of the first and second kind, respectively, and

\[
H_v^{(1)}(e^{j\pi z}) = -e^{-j\pi v} H_v^{(2)}(z),
\]

(3.12)
in the original equations. Using the notations used in the previous section, all functions of interest can be written in terms of a general form in (3.1) as

\[
\mathcal{F} = \int_0^\infty f(\xi_1, \cdots, \xi_n) g(\lambda) J_i(\lambda \rho) d\lambda,
\]

(3.13)

where \( i \) denotes the order of the Bessel function which is either 0 or 1, and \( f(\xi_1, \cdots, \xi_n) \) is a function of \( \xi_k, k = 1, \cdots, n \), or \( \lambda^2 \), i.e., an even function of \( \lambda \). For all functions associated with either a horizontal or vertical electric current source, or a horizontal magnetic current
source, where \( g(\lambda) \) is an odd function of \( \lambda \), and \( i = 0 \), i.e., Bessel function of zeroth order, it can be shown that

\[
F \{ HE, VE, HM \} = \int_0^\infty f(\xi_1, \cdots, \xi_n)g(\lambda)J_0(\lambda \rho)\,d\lambda
\]

\[
= \int_0^\infty f(\xi_1, \cdots, \xi_n)g(\lambda)\frac{1}{2}[H_0^{(1)}(\lambda \rho) + H_0^{(2)}(\lambda \rho)]\,d\lambda
\]

\[
= \frac{1}{2} \int_0^\infty f(\xi_1, \cdots, \xi_n)g(\lambda)H_0^{(1)}(\lambda \rho)\,d\lambda + \frac{1}{2} \int_0^\infty f(\xi_1, \cdots, \xi_n)g(\lambda)H_0^{(2)}(\lambda \rho)\,d\lambda,
\]

(3.14)

whose first integral in the right-hand side can be rewritten as

\[
\frac{1}{2} \int_0^\infty f(\xi_1, \cdots, \xi_n)g(\lambda)H_0^{(1)}(\lambda \rho)\,d\lambda = \frac{1}{2} \int_0^{-\infty} f(\xi_1, \cdots, \xi_n)g(-\lambda')H_0^{(1)}(-\lambda' \rho)(-d\lambda')
\]

\[
= \frac{1}{2} \int_{-\infty}^0 f(\xi_1, \cdots, \xi_n)g(\lambda')H_0^{(2)}(\lambda' \rho)\,d\lambda.
\]

(3.15)

Therefore, (3.14) becomes

\[
F \{ HE, VE, HM \} = \frac{1}{2} \int_{-\infty}^\infty f(\xi_1, \cdots, \xi_n)g(\lambda)H_0^{(2)}(\lambda \rho)\,d\lambda.
\]

(3.16)

Likewise, for functions associated with a horizontal magnetic current source, where \( g(\lambda) \) is an even function and \( i = 1 \), i.e., Bessel function of first order, it can be shown that

\[
F^{HM} = \int_0^\infty f(\xi_1, \cdots, \xi_n)g(\lambda)J_1(\lambda \rho)\,d\lambda
\]

\[
= \frac{1}{2} \int_0^\infty f(\xi_1, \cdots, \xi_n)g(\lambda)H_1^{(1)}(\lambda \rho)\,d\lambda + \frac{1}{2} \int_0^\infty f(\xi_1, \cdots, \xi_n)g(\lambda)H_1^{(2)}(\lambda \rho)\,d\lambda.
\]

(3.17)

The first term in the right-hand side can be rewritten as

\[
\frac{1}{2} \int_0^\infty f(\xi_1, \cdots, \xi_n)g(\lambda)H_1^{(1)}(\lambda \rho)\,d\lambda = \frac{1}{2} \int_0^{-\infty} f(\xi_1, \cdots, \xi_n)g(\lambda)H_1^{(1)}(-\lambda' \rho)(-d\lambda')
\]

\[
= \frac{1}{2} \int_{-\infty}^0 f(\xi_1, \cdots, \xi_n)g(\lambda')H_1^{(2)}(\lambda' \rho)\,d\lambda',
\]

(3.18)
thus (3.17) becomes

\[ F^H = \frac{1}{2} \int_{-\infty}^{\infty} f(\xi_1, \ldots, \xi_n) g(\lambda) H^{(2)}_1(\lambda \rho) d\lambda. \] (3.19)

Now, introducing a new function \( T(\lambda, \xi_1, \ldots, \xi_n) \) such that \( k_0^2 T(\lambda, \xi_1, \ldots, \xi_n) \lambda = f(\xi_1, \ldots, \xi_n) g(\lambda) \), then (3.16) and (3.16) can be expressed in a more compact form as

\[ F = \frac{1}{2k_0^2} \int_{-\infty}^{\infty} T(\lambda, \xi_1, \ldots, \xi_n) H^{(2)}_i(\lambda \rho) \lambda d\lambda, \] (3.20)

where \( i \) is either 0 or 1 as before. Since the integrand of (3.20) includes a branch cut factor \( \sqrt{k_0^2 - \lambda^2} \), it is preferable to eliminate this branch cut by using the change of variable \( \lambda = k_0 \sin \gamma \). Then (3.20) becomes

\[ F = \frac{1}{2} \int_C T(\lambda, \xi_1, \ldots, \xi_n) H^{(2)}_i(k_0 \rho \sin \gamma) \sin \gamma \cos \gamma d\gamma. \] (3.21)

The large argument approximation of \( H^{(2)}_0 \) and \( H^{(2)}_1 \) can be given by

\[ H^{(2)}_0(z) = \sqrt{\frac{2}{\pi z}} e^{-j(z-\frac{\pi}{4})} (1 + \frac{j}{8z} - \frac{9}{128z^2}), \] (3.22)

and

\[ H^{(2)}_1(z) = \sqrt{\frac{2}{\pi z}} e^{-j(z-\frac{3\pi}{4})} (1 - \frac{j}{8z} + \frac{15}{128z^2}), \] (3.23)

respectively, with \( \text{Im} \sqrt{z} < 0 \). Using (3.22) and (3.23) in (3.20), it can be seen that the integral for \( F \) has a saddle point at \( \gamma_s = \pi/2 \). Replacing the integration path \( C \) by the steepest descent path (SDP) at \( \gamma_s \) and taking into account the poles crossed by sweeping \( C \) to SDP yields

\[ F = -j\pi \sum_p R^T_p H^{(2)}_i(k_0 \rho \sin \gamma_p) \sin \gamma_p \cos \gamma_p \]

\[ + \left\{ \frac{e^{j\pi/4}}{\sqrt{2\pi k_0 \rho}} \right\} \left( 1 + \frac{j}{8k_0 \rho} - \frac{9}{128k_0^2 \rho^2} \right) \int_{SDP} T e^{-j k_0 \rho \sin \gamma \sqrt{\sin \gamma \cos \gamma} d\gamma. \] (3.24)
where \( R_p^T \) denotes the residue of \( T \) at the \( p^{th} \) pole given by

\[
R_p^T = \lim_{\gamma \to \gamma_p} (\gamma - \gamma_p) T = \frac{N(\gamma)}{\partial D(\gamma)} |_{\gamma = \gamma_p}.
\]  

Figure 3.3 shows the original Sommerfeld integration contour in the \( \lambda \)-plane while figure 3.4 shows the integration path \( C \) and \( SDP \) in the \( \gamma \)-plane.

Figure 3.3: The original Sommerfeld integration contour in the \( \lambda \)-plane.

Now, let \( I_a \) denote the integral along \( SDP \) in (3.24), i.e.,

\[
I_a = \int_{SDP} T e^{-j k_0 \rho \sin \gamma} \sqrt{\sin \gamma \cos \gamma} d\gamma.
\]  

Using the following change of variable

\[
\sin \gamma = 1 - j s^2,
\]

\[
\cos \gamma = -s \sqrt{2j + s^2},
\]

\[
\cos \gamma d\gamma = -2j s ds,
\]

\[\text{(3.27)} \]  
\[\text{(3.28)} \]  
\[\text{(3.29)} \]
Figure 3.4: Integration paths $C$ and $SDP$ in complex $\gamma$-plane

in (3.26) yields

$$I_a = -2je^{-jka}\int_{-\infty}^{\infty} T\sqrt{1 - js^2e^{-ka\rho s^2}}s ds. \quad (3.30)$$

The function $T$ in the above equation is analytic in the neighborhood of the origin ($s = 0$) except at some surface and leaky wave poles given respectively by

$$s_p = \pm e^{-j\pi/4}\sqrt{\sin \gamma_p - 1}, \quad (3.31)$$

where $-\pi/2 < \arg(\sqrt{\sin \gamma_p - 1}) < 0$.

For large separations, the main contribution of $I_a$ comes from $s$ close to the origin (which is the saddle point). The integrand can be approximated by the expansion around
the origin using the method of Laurant expansion discussed in [65] as

\[ G(s) = Ts\sqrt{1 - js^2} \approx \sum_p \frac{r_p^T}{s - s_p} + \sum_{n=0}^{N} a_n s^n. \] (3.32)

The terms with odd values of \( n \) in (3.32) will be odd functions of \( s \) and thus will not contribute to \( I_a \). Also, typically only \( N = 2 \) will give reasonably accurate results. The coefficients for each term in (3.32) can be given by

\[ r_p^T = \lim_{s \to s_p} (s - s_p)G(s) \]
\[ = s_p \sqrt{1 - js_p^2} \frac{N(\gamma)}{\partial \gamma} \bigg|_{\gamma = \gamma_p, s = s_p} \]
\[ = \frac{j}{2} \sqrt{\sin \gamma_p \cos \gamma_p} R_p^T, \] (3.33)

\[ a_0 = \left[ G(s) - \sum_p \frac{r_p^T}{s - s_p} \right] \bigg|_{s=0} \]
\[ = \sum_p \frac{r_p^T}{s_p}, \] (3.34)

and

\[ a_2 = \frac{1}{2} \frac{\partial^2}{\partial s^2} \left[ G(s) - \sum_p \frac{r_p^T}{s - s_p} \right] \bigg|_{s=0} \]
\[ = \sum_p \frac{r_p^T}{s_p^3} + \left. \frac{\partial T}{\partial s} \right|_{s=0}, \] (3.35)

where \( R_p^T \) in (3.33) denotes the residue of \( T \) given in (3.25).

Substituting the integrand in (3.30) with the approximation of \( G(s) \) given in (3.32) with the coefficients given above, and integrating over \( s \) yields the following asymptotic approximation to \( I_a \) as

\[ I_a \approx -2j e^{-jk_0} \sqrt{\frac{\pi}{k_0 \rho}} \left\{ \sum_p \frac{r_p^T}{s_p} \left[ 1 - F(jk_0 \rho s_p^2) \right] + \frac{1}{2k_0 \rho} \left[ \sum_p \frac{r_p^T}{s_p^3} + \left. \frac{\partial T}{\partial s} \right|_{s=0} \right] \right\}, \] (3.36)
where the transition function $F(x)$ is given by [66, 67]:

$$F(x) = 2j\sqrt{x}e^{jx} \int_{\sqrt{x}}^{\infty} e^{-jt^2} dt, \quad -\frac{3\pi}{4} < \arg \sqrt{x} < \frac{\pi}{4}. \quad (3.37)$$

Noticing that $T$ is a function of $\lambda$ and $\xi_i, i = 1, \cdots, n$, where $\xi_n = \sqrt{k_0^2 - \lambda^2} = k_0 \cos \gamma = -k_0 s \sqrt{2j + s^2}.$ and $s = 0$ implies $\xi_n = 0,$ $\left. \frac{\partial T}{\partial s} \right|_{s=0}$ in (3.36) can be found from

$$\left. \frac{\partial T}{\partial s} \right|_{s=0} = \left. \frac{\partial T}{\partial \xi_n} \frac{\partial \xi_n}{\partial s} \right|_{s=0} = -\sqrt{2} e^{j\pi/4} k_0 \left. \frac{\partial T}{\partial \xi_n} \right|_{\xi_n=0}. \quad (3.38)$$

Thus, $\frac{\partial T}{\partial \xi_n}$ can be easily evaluated. This allows this asymptotic approximation to be applicable to evaluate the multilayered media Green’s function.

It is noted that when some of the poles are extremely close to the saddle point, accurate $a_2$ may not be obtained due to a finite precision used in the calculation. Therefore, it is desirable to avoid the cases where there is a pole extremely close to $s = 0$ or $\lambda = k_0.$ It is found in this study that this method is still valid even when $|s_p|$ is as small as $10^{-4}.$ However, considering each term in (3.24), the terms associated with surface wave and leaky wave poles have an algebraically decaying rate of $1/\sqrt{\rho},$ while the term associated with $[1 - F(jk_0\rho s_p^2)]$ of $I_a$ in (3.36) has a decaying rate of $1/\rho$ but $[1 - F(jk_0\rho s_p^2)] \rightarrow 0$ for large $\rho,$ and the second term of $I_a$ has a decaying rate of $1/\rho^2.$ Thus, $I_a$ will become negligibly small when $\rho$ is large, and only surface and leaky wave terms remain dominant. Hence, the asymptotic closed-form approximation is expected to be accurate for large $\rho$ even in cases where accurate $a_2$ cannot be obtained, which makes the method considerably reliable.
3.4 Numerical Results and Discussion

In this section, some numerical results regarding multilayered media dyadic Green’s function, which are obtained by the methods mentioned in sections 3.1 and 3.3, are presented. The geometry used in all calculations is shown in figure 3.5 and the frequency used here is 30 GHz. All functions are evaluated by both numerical integration method and asymptotic approximation to illustrate the validity of both methods.

\[
\begin{align*}
Z_5 &= 1.8 & \varepsilon_r^5 &= \varepsilon_0 \\
Z_4 &= 1.1 & \varepsilon_r^4 &= 2.1 \\
Z_3 &= 0.8 & \varepsilon_r^3 &= 12.5 \\
Z_2 &= 0.3 & \varepsilon_r^2 &= 9.8 \\
Z_1 &= 0.0 & \varepsilon_r^1 &= 8.6
\end{align*}
\]

Figure 3.5: Configuration of a grounded multilayered medium used in the calculations. The unit of thickness is mm and the arrow denotes the location of the horizontal current source.

As mentioned earlier, the surface wave and leaky wave poles need to be first extracted to be used in later calculations. For this geometry, using the method mentioned in section 3.2 yields one TM surface wave at \(2.4354k_0\), one TE surface wave pole at \(1.7365k_0\) and one TM leaky wave pole at \(1.0092k_0\). The surface wave pole locations agree with those reported in [63]. These poles are used to compute the residue for each function described in chapter 2, which is in turn used in both numerical integration method and asymptotic closed-form approximation. Therefore, obtaining correct poles and their residues is crucial in the methods developed in this work.
For all functions, the numerical integration method is used to compute for separations $0 \leq \rho \leq 1.5\lambda_0$, while the asymptotic approximation is used when $.5\lambda_0 \leq \rho \leq 3.2\lambda_0$. Figure 3.6 shows the results of $U^{HE}$ and $W^{HE}$ functions assuming that a horizontal source is located in the fourth layer, i.e., at $z = z_4$ as shown in the figure 3.5. It is also noted that $U^{HE}$ and $W^{HE}$ are for calculating the tangential components of the electric field at $z = z_4$, due to the horizontal electric source located at $z = z_4$. As can be seen from the figure, the results obtained from the asymptotic approximation agree very well with those from the numerical integration method when $\rho > \lambda_0$, since at frequency 30 GHz the surface wave components will be dominant for large $\rho$ and thus the approximation becomes quite accurate.

Likewise, figure 3.7 shows the results of $V^{VE}_1$ and $V^{VE}_1$ functions, which are used to compute the vertical component of the electric field in the first layer due to a vertical electric source in the first layer. Same trend as that of $U^{HE}$ function can be observed here. Also, figure 3.8 shows the results of $V^{VE}_2$ and $V^{VE}_2$ functions, which are used to compute the normal component of the electric field in the second layer due to a vertical electric source in the same layer. Figures 3.9 and 3.10 show results of $V^{VE}_1$, $V^{VE}_1$, $V^{VE}_2$, and $V^{VE}_2$ functions, which are used to compute the normal component of the electric field due to a vertical electric source when the source and the observation are in different layers. The last results regarding the fields due to vertical electric sources shown in figure 3.11 are plots of $W^{VE}_1$ and $W^{VE}_2$ functions, which are used to compute the tangential components of the electric field at $z = z_4$ due to vertical sources in the first and second layers, respectively.

The results of functions used to represent the electric field due to a horizontal magnetic source on the ground plane are shown in figures 3.12 and 3.13. Figure 3.12 shows the results of $U^{HM}$ and $W^{HM}$ functions, which are used to compute the tangential components
of the electric field at $z = z_4$, while figure 3.13 shows the results of $V_{1}^{HM}$ and $V_{3}^{HM}$, which are used to compute the normal component of the electric field in the first and third layer, respectively.

As can be seen from all results, the asymptotic closed-form approximation gives good approximated results of all functions for large $\rho$, which makes it extremely efficient when dealing with very large array problems, because it can significantly reduce the MoM operator matrix filling time.

Finally, it is worthwhile comparing the method developed in this work, namely, the combination of numerical integration method and asymptotic approximation, with the the DCIM method, which is an alternative approach to evaluate the multilayered media Green’s functions, when applied in the method of moments for large electromagnetic problems. The approach used in this work has the following advantages:

- The computational cost is relatively small, since it includes only simple numerical integration and closed-form evaluation of integrands.
- The results from the asymptotic approximation consist only a few terms, thus they are highly efficient for large electromagnetic problems.
- The asymptotic approximation can also be used to verify the results from the numerical integration, thus providing more reliability to the approach.

and the following disadvantages:

- The asymptotic evaluation is valid only when $\rho$ is larger than a certain distance, thus the numerical integration is required to compute the functions for the near-field region.
• The results from the numerical integration have to be tabulated and interpolations are thus required to obtain the value for any specific \( \rho \).

On the other hand, the DCIM method has the following advantages:

• The final results are in closed-form spatial Green’s functions, thus no function table is constructed and no interpolation is required.

• It can be applied to both near-field and far-field regions.

and the following disadvantages:

• The approximation process can be computationally expensive, especially when integrands become fast-varying functions. This happens when the sampling path is close to a singular point, i.e., a pole or a branch point.

• It generally requires significantly more terms than the asymptotic evaluation, thus it is less efficient for large electromagnetic problems.

• The effects of complex images can lead to instability in the far-field region if the pole extraction is not done properly.
Figure 3.6: Plot of $U^{HE}$ and $W^{HE}$ functions for the geometry shown in figure 3.5. The horizontal electric source is located at $z = 1.1$ mm
Figure 3.7: Plot of $V_{1a}^{VE}$ and $V_{1b}^{VE}$ functions for the geometry shown in figure 3.5 when $|z - z'| = 0.1125$ mm and $(z + z') = 0.6$ mm, respectively.
Figure 3.8: Plot of $V_{2a}^{VE2}$ and $V_{2b}^{VE2}$ functions for the geometry shown in figure 3.5 when $|z - z'| = 0.136$ mm and $(z + z') = 1.1$ mm, respectively.
Figure 3.9: Plot of $V_{2a}^{VE1}$ and $V_{2b}^{VE1}$ functions for the geometry shown in figure 3.5 when $z' = 0.075$ mm and $z = 0.391$ mm, respectively
Figure 3.10: Plot of $V_{3a}^{VE2}$ and $V_{3b}^{VE2}$ functions for the geometry shown in figure 3.5 when $z' = 0.436$ mm and $z = 0.875$ mm, respectively.
Figure 3.11: Plot of $W^{VE1}$ and $W^{VE2}$ functions for the geometry shown in figure 3.5 when $z' = 0.0$ mm and $z' = 0.391$ mm, respectively.
Figure 3.12: Plot of $U^{HM}$ and $W^{HM}$ functions for the geometry shown in figure 3.5. The horizontal magnetic source is located on the ground plane.
Figure 3.13: Plot of $V_1^{HM}$ and $V_3^{HM}$ functions for the geometry shown in figure 3.5 when $z = 0.15$ mm and $z = 0.875$ mm, respectively.
CHAPTER 4

INTEGRAL EQUATION FORMULATION AND IMPLEMENTATION OF METHOD OF MOMENTS

In this chapter, an integral equation will be formulated to solve for the unknown electric field radiated or scattered by finite antenna arrays of interest. The general array geometry is shown in figure 4.1 and the array elements can be either dipole or patch antenna, as shown in the figure. It is noted that patch antennas can be excited by either microstrip-line, coaxial probe, or aperture. By applying the equivalence theorem and appropriate boundary conditions, an integral equation governing the antenna array can be obtained in which the unknown electric field can be given in terms of unknown equivalent currents. Once the integral equation is obtained, the method of moments (MoM) can be applied by first representing unknown currents in terms of known expansion functions with unknown coefficients, and then the Galerkin testing can be performed to convert integral equations into matrix equations to be solved for unknown coefficients.

This chapter consists of three major parts. In section 4.1 the general approach to formulate integral equations via the equivalence theorem will be discussed, followed by the details of basis functions used for expanding the unknown currents in section 4.2. Finally, section 4.3 will explain the method of moments procedure and the evaluations of elements in the MoM operator matrix and the excitation vector.
4.1 Integral Equation

Consider a finite periodic planar array of perfectly conducting microstrip antenna elements in a grounded multilayered medium as depicted in figure 4.1. The array elements can be either on the topmost layer or embedded between layers and the planar grounded multilayered medium is assumed to be of infinite extent and to be surrounded by free space.

![Figure 4.1: A finite antenna array in a multilayered medium](image)

It is assumed that the ground plane and all parts of antenna elements are made of a perfect conductor. By applying the surface equivalence theorem the array elements can be replaced by equivalent electric currents radiating in the same multilayered structure. Now, let \( J_{nm} \) denote the \((n, m)^{th}\) equivalent electric current for the \((n, m)^{th}\) antenna array.
element, then the electric field radiated from the entire array in the \( i^{th} \) layer, denoted here by \( E_s^i \), can be given by:

\[
E_s^i = \sum_{n=1}^{N} \sum_{m=1}^{M} \int_{S_{nm}} \mathbf{G}(\bar{r}|\bar{r}^\prime_{nm}) \cdot \mathbf{J}_{nm}(\bar{r}^\prime_{nm}) d\bar{s}^\prime,
\]

(4.1)

where \( \mathbf{G}(\bar{r}|\bar{r}^\prime) \) is the appropriate dyadic Green’s function for the electric field due to an electric point source and \( N, M \) is the number of elements in the \( x \) and \( y \) directions, respectively. Also, \( S_{nm} \) denotes the surface on which the \( nm^{th} \) source is located.

The electric field integral equation (EFIE) representing the boundary condition that the total tangential electric field must vanish on the perfectly conducting array elements can be written as:

\[
-\hat{n} \times E_s = \hat{n} \times E^{inc} \quad \text{on array elements},
\]

(4.2)

where \( \hat{n} \) denotes the unit vector normal to the array elements, and \( E_s \) and \( E^{inc} \) denote the radiated electric field and the incident electric field, respectively. For the radiation problem, the incident electric field is the field radiated by the local sources which model the element excitations. In this work, a delta gap generator is used to model an excitation source for a printed dipole, while an equivalent vertical current ribbon is used to represent the local source for a microstrip-line fed patch on a single layer substrate. It can be given by [11]

\[
\mathbf{J}^i = \hat{z} \sqrt{W_e} \delta(x - x_p) \delta(y - y_p) \quad \text{for} \ 0 \leq z \leq d,
\]

(4.3)

where

\[
\begin{align*}
(x_p, y_p) &= \text{the feed location} \\
W_e &= \text{the effective width given by} \\
W_e &= W + 0.412 \frac{\epsilon_e + 0.3}{\epsilon_e - 0.258} \frac{W + 0.262d}{W + 0.813d} d \\
W &= \text{the width of the microstrip feed line} \\
d &= \text{the substrate thickness} \\
\epsilon_e &= \text{the effective dielectric constant given by} \\
\epsilon_e &= \frac{\epsilon_e + 1}{2} + \frac{\epsilon_e - 1}{2} \left(1 + \frac{10d}{W}\right)^{-1/2} \\
\epsilon_r &= \text{the dielectric constant of the substrate}.
\end{align*}
\]
Using this current ribbon model, the incident electric field can then be given by

$$E^{\text{inc}}(\bar{r}) = \sum_{n=1}^{N_p} J_n \int_{S_n} \overline{G}^{VE}(\bar{r}|\bar{r}_n') \cdot J_i^n(\bar{r}_n') \, ds',$$

(4.4)

where $\overline{G}^{VE}(\bar{r}|\bar{r}')$ denotes the dyadic Green’s function for the electric field due to the vertical electric current source, $J_n$ denotes the driving current for the $n^{th}$ patch antenna and $N_p$ is the total number of microstrip lines.

Likewise, for a coaxial probe-fed patch antenna array, a magnetic frill generator is used to model an excitation from the aperture formed by a coaxial probe as shown in figure 4.2. Here, the coaxial aperture is replaced by an equivalent magnetic current with the aperture closed. It is given by[56]

$$M^f(\rho) = E_f \times \hat{z} = -\hat{\phi} \frac{V^i}{\rho \ln(b/a)} \quad a \leq \rho \leq b,$$

(4.5)

where $a, b$ denote the inner and outer of the probe aperture, respectively, and $V^i$ is the driving voltage. Using this magnetic frill generator mode, the incident electric field can then be given by

$$E^{\text{inc}}(\bar{r}) = \sum_{n=1}^{N_p} V_n \int_{S_n} \overline{G}^{HM}(\bar{r}|\bar{r}_n') \cdot M^f_n(\bar{r}_n') \, ds',$$

(4.6)

where $\overline{G}^{HM}(\bar{r}|\bar{r}')$ denotes the dyadic Green’s function for the electric field due to the horizontal magnetic current source, $V_n$ denotes the driving voltage for the $n^{th}$ probe and $N_p$ is the total number of probes.

A similar magnetic current source can also be used to model the source for an aperture-coupled patch antenna. Since the aperture is assumed to be thin and short compared to the wavelength, it can be replaced by an equivalent magnetic current on the ground plane, which can be given by

$$M_p^i(\xi, \zeta) = \xi \sin k_\epsilon \left( \frac{1}{2} - |\xi - \xi_p| \right) \frac{w}{w \sin \frac{k_\epsilon l}{2}}, \quad |\xi - \xi_p| < \frac{l}{2}, |\zeta - \zeta_p| < \frac{w}{2},$$

(4.7)
where \( l, w \) denote the length and width, respectively, of the aperture, and \( k_e \) is a proper wave number. \( \xi, \zeta \) is the coordinates shown in figure 4.3 and \( \xi_p, \zeta_p \) denote the center of the aperture.

Similar procedures can be applied to the scattering problem by using the electric field induced by the plane wave which impinges upon the topmost layer as the incident electric field. It is noted that the incident field consists of both incident and reflected waves, which are found by enforcing boundary conditions at each layer boundary including the ground plane. Table 4.1 summarizes the models used to represent each excitation method in this work. Note that the plane wave incidence is used in all scattering problems regardless of element type.

![Diagram of a coaxial probe and its corresponding magnetic frill generator](image)

Figure 4.2: A coaxial probe and its corresponding magnetic frill generator
Table 4.1: Excitation models

<table>
<thead>
<tr>
<th>Array Element Type</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dipole radiator</td>
<td>Delta gap generator</td>
</tr>
<tr>
<td>Microstrip-line fed patch radiator</td>
<td>Vertical current ribbon</td>
</tr>
<tr>
<td>Probe-fed patch radiator</td>
<td>Magnetic frill generator</td>
</tr>
<tr>
<td>Aperture-coupled patch radiator</td>
<td>Magnetic frill generator</td>
</tr>
<tr>
<td>All element types for scattering case</td>
<td>Plane wave incidence</td>
</tr>
</tbody>
</table>

Figure 4.3: Coupling aperture geometry

4.2 Basis Functions

As mentioned earlier, in order to solve for unknown currents in the integral equation discussed in the previous section, the currents are expanded in terms of known expansion functions and associated unknown coefficients. Good basis functions should well represent the currents to be solved. For elements consisting of only simple shapes, e.g., wire probes, rectangular strips and patches, piecewise sinusoidal (PWS) basis functions are considered efficient and widely used, and thus in this work they are extensively used as the basis
functions. The basis functions in this work can be categorized according to their geometries into three groups, namely, patch modes, wire modes and a wire to plate attachment mode, which correspond to patch, probe and wire to plate attachment currents, respectively. Since, in general, the currents on each array element can have both \( \hat{x} \) and \( \hat{y} \) components, their expansions consist of both \( \hat{x} \)-directed and \( \hat{y} \)-directed patch modes. For the \( \hat{x} \)-directed patch mode, the basis function can be given by

\[
\mathbf{f}^x(x, y) = \hat{x} f^x(x, y) = \frac{\sin k_e (h - |x - x_i|)}{w \sin k_e h}, \quad |x - x_i| < h, |y - y_i| < \frac{w}{2}, \quad (4.8)
\]

where \( h, w \) denote the half-length and the width, respectively, of the expansion mode, \((x_i, y_i)\) denotes the coordinates of the center of the mode, and \( k_e \) is the suitable wavenumber chosen to be the effective propagation constant of an infinite strip transmission line of width \( w \) located in the same medium. Likewise, the basis function for \( \hat{y} \)-directed patch mode is given by

\[
\mathbf{f}^y(x, y) = \hat{y} f^y(x, y) = \frac{\sin k_e (h - |y - y_i|)}{w \sin k_e h}, \quad |y - y_i| < h, |x - x_i| < \frac{w}{2}. \quad (4.9)
\]

These basis functions are depicted in figure 4.4 where circles and diamonds in the figure denote the center points of the expansion modes for \( \hat{x} \)-directed and \( \hat{y} \)-directed patch modes, respectively. For the \( \hat{x} \)-directed patch modes, \((h, w)\) are chosen to be \((w_y, w_x)\), while \((h, w)\) for the \( \hat{y} \)-directed patch modes are chosen to be \((w_x, w_y)\).

Similarly, probe currents are expanded using a piecewise sinusoidal function on a cylinder which models the probe; they are given by:

\[
\mathbf{f}^w(x, y, z) = \hat{z} f^w(x, y, z) = \begin{cases} 
\frac{\sin k_e (h - z)}{2 \pi a \sin k_e h}, & 0 < z < h, x^2 + y^2 = a^2 \quad \text{first wire mode}, \\
\frac{\sin k_e (h - |z - z_i|)}{2 \pi a \sin k_e h}, & |z - z_i| < h, x^2 + y^2 = a^2 \quad \text{otherwise}, 
\end{cases}
\]

where \( z_i \) denotes the \( z \) coordinate of the center of the \( i^{th} \) wire mode, \( h \) is the length of each wire segment, and \( a \) denotes the radius of the probe wire. It is noted that wire segments
Figure 4.4: Basis functions used to expand patch currents

may have different length due to the differences in the thickness of layers. This function is shown in figure 4.5. As can be seen from the figure, the first wire mode has only one segment while the other modes have two. Note that $z = 0$ is at the base of the probe; i.e., at the level of the ground plane.

Finally, a special attachment mode function is also introduced to enforce the continuity of the current at the probe-patch junction. It consists of two parts, namely, a probe part and a patch part. The basis function for the probe part is the same as the one used to represent the wire modes but has only one segment and it is given by

$$f_{aw}^z(x, y, z) = z f_{aw}^x(x, y, z) = \frac{z \sin k_e(z - z_a)}{2\pi a \sin k_e h}, \quad z_a < z < z_m, x^2 + y^2 = a^2, \quad (4.11)$$
where $z_a$ denotes the lower end of the probe part attachment mode and $z_m$ is the $z$-coordinate of the $m^{th}$ layer, i.e., the layer on which the patch antennas are located. It is also shown in figure 4.5. For the probe location away from the patch edge to a certain distance, the patch part attachment mode is represented by a planar circular disk monopole lying on the patch as shown in figure 4.6(a). However, when the probe comes closer to the patch edge, it has to be modified to account for the current behavior near the edge as shown in figure 4.6(b). The basis function for the patch part attachment mode used in this work is adapted from the one implemented by Pozar et.al.[25] for a monopole mounted near an edge; it is given by:

$$f_{ap}(x, y, z) = \hat{\rho} f_{ap}(x, y, z) = \hat{\rho} A \frac{\sin k_e [b(\phi) - \rho]}{2\pi \rho \sin k_e [b(\phi) - a]} [\sin k_e b(\phi)]^{1.2}, \quad \rho^2 = x^2 + y^2,$$

(4.12)
where $x_p$ denotes probe location, $b$, $a$ denote the inner, outer radius of the disk, respectively, and $A$ is a suitable normalization constant chosen for 1 A terminal current. Also, the azimuthal variation $b(\phi)$ is given by

$$b(\phi) = \begin{cases} b_c \text{ (constant)} & \text{for } |\phi| \leq \phi_0, \\
\frac{x_p}{\cos(\pi - |\phi|)} & \text{otherwise,}
\end{cases}$$

(4.13)

for $0 \leq |\phi| \leq \pi$ and where $\phi_0 = \pi - \cos^{-1}(x_p/b_c)$. The 1.2 power is chosen such as to ensure the convergence of the self-reactance of a short PWS monopole of length $d$, which diverges as $\ln kd/d$ as $d$ becomes very small[24].

![Diagram](image)

(a) $x_p > b_c$

(b) $x_p < b_c$

Figure 4.6: Basis function for the patch-part attachment mode current

Since the basis functions are used to expand the unknown currents, which depend on the type of array element and the excitation method, the number and type of basis functions used for each element can be quite different. For example, only $\hat{x}$-directed patch modes are used in the dipole array case for both radiation and scattering problems, since the width of dipole is assumed to be thin and thus $\hat{y}$-directed currents are not excited, while all basis functions are required in the analysis of probe-fed patch antenna arrays. Table 4.2 summarizes basis functions used in each problem in this work. It is noted that only $\hat{x}$-directed
Table 4.2: Basis functions used in this work

<table>
<thead>
<tr>
<th>Problem</th>
<th>$\hat{x}$-patch</th>
<th>$\hat{y}$-patch</th>
<th>wire</th>
<th>attachment</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dipole array</td>
<td>✓</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Microstrip-line fed patch array: Radiation</td>
<td>✓</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Microstrip-line fed patch array: Scattering</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Probe-fed patch array</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Aperture-coupled patch array</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

patch modes are used in the radiation problem of the microstrip-line fed patch array because only $\hat{x}$-directed currents are dominant, while both $\hat{x}$-directed and $\hat{y}$-directed patch modes are required in the scattering problem, since incident plane waves can excite both $\hat{x}$-directed and $\hat{y}$-directed currents.

4.3 MoM Implementation

Once the basis functions used to expand all currents are determined, the radiated fields due to each current can be given in terms of the basis functions and unknown coefficients as follows

$$E^s(\vec{r}) = \sum_{n=1}^{N_x} i^n_x \int_{S_n} \overline{G}^{HE}(\vec{r}|\vec{r}_n') \cdot f^n_x(\vec{r}_n')ds' + \sum_{m=1}^{N_y} i^y_m \int_{S_m} \overline{G}^{HE}(\vec{r}|\vec{r}_m') \cdot f^y_m(\vec{r}_m')ds' + \sum_{p=1}^{N_w} i^w_p \int_{S_p} \overline{G}^{VE}(\vec{r}|\vec{r}_p') \cdot f^w_p(\vec{r}_p')ds' + \sum_{q=1}^{N_a} i^a_q \left( \int_{S_{qw}} \overline{G}^{HE}(\vec{r}|\vec{r}_{q,w}') \cdot f^{aw}_{q,w}(\vec{r}_{q,w}')ds' + \int_{S_{qp}} \overline{G}^{HE}(\vec{r}|\vec{r}_{q,p}') \cdot f^{ap}_{q,p}(\vec{r}_{q,p}')ds' \right)$$  

(4.14)

where $\overline{G}^{HE}(\vec{r}|\vec{r}')$ and $\overline{G}^{VE}(\vec{r}|\vec{r}')$ denote the dyadic Green's functions for the electric field due to the horizontal electric current source and vertical electric current source, respectively. Also, $N_x, N_y, N_w, N_a$ denote the total number of $\hat{x}$-directed patch modes, $\hat{y}$-directed
patch modes, wire modes and attachment modes, respectively. \( i^\alpha_p \) \( \alpha \in \{ x, y, w, a \} \) denotes the unknown coefficient associated with the \( p^{th} \) expansion mode of each current. \( S_n^\beta \) with \( \beta \in \{ x, y, w, aw, ap \} \) denotes the support area of the \( n^{th} \) \( \beta \) mode, e.g., \( S_n^w \) is the area of the \( n^{th} \) wire mode. Note that \( aw \) and \( ap \) denote the area of the wire part and patch part, respectively, of the attachment mode. It is noted in the previous section that all basis functions are required only when array elements are probe-fed patch antennas. Table 4.3 summarizes the number of total unknowns, denoted here by \( N_{total} \), for antenna arrays of different array elements studied in this work.

The electric field integral equation obtained by enforcing the boundary condition which requires that the total tangential electric field must vanish on the array elements can then be written as:

\[-\hat{n} \times E^s = \hat{n} \times E^{inc} \text{ on expansion modes,} \quad (4.15)\]

where \( \hat{n} \) denotes the unit vector normal to the expansion modes, and \( E^s \) and \( E^{inc} \) denote the scattered and incident electric fields, respectively.

Applying the Galerkin testing, by using the expansion functions as the testing functions, to the integral equation in (4.15) yields the following matrix equation:

\[ Z_i = v \quad (4.16) \]
where

\[
Z = \begin{bmatrix}
Z^{xx} & Z^{xy} & Z^{xw} & Z^{xa} \\
Z^{yx} & Z^{yy} & Z^{yw} & Z^{ya} \\
Z^{wx} & Z^{wy} & Z^{ww} & Z^{wa} \\
Z^{ax} & Z^{ay} & Z^{aw} & Z^{aa}
\end{bmatrix},
\]

(4.17)

\[
v = \begin{bmatrix}
v^x \\
v^y \\
v^w \\
v^a
\end{bmatrix},
\]

(4.18)

and

\[
i = \begin{bmatrix}
i^x \\
i^y \\
i^w \\
i^a
\end{bmatrix},
\]

(4.19)

where \(\mathbf{i}^{x} = [i_{1}^{x}, i_{2}^{x}, \ldots, i_{N_{x}}^{x}]^{T}\), \(\mathbf{i}^{y} = [i_{1}^{y}, i_{2}^{y}, \ldots, i_{N_{y}}^{y}]^{T}\), \(\mathbf{i}^{w} = [i_{1}^{w}, i_{2}^{w}, \ldots, i_{N_{w}}^{w}]^{T}\) and \(\mathbf{i}^{a} = [i_{1}^{a}, i_{2}^{a}, \ldots, i_{N_{a}}^{a}]^{T}\). It is noted that by using the expansion functions as the testing functions, \(Z\) will become symmetric, which is assumed throughout this work.

The submatrices \(Z^{\alpha\beta}\) with \(\alpha, \beta \in \{x, y, w, a\}\) represent the couplings between the \(\alpha\) expansion modes and the electric field due to the \(\beta\) expansion modes, e.g., \(Z^{xy}\) represents the couplings between the \(\hat{x}\)-directed patch modes and the electric field due to the \(\hat{y}\)-directed patch modes. Using the Green’s functions developed in the previous chapter, the elements in each submatrices \(Z^{xx}, Z^{yy}\) and \(Z^{yw}\) are given by

\[
Z_{p,q}^{xx} = - \int_{S_{p}^{x}} \int_{S_{q}^{x}} f_{p}^{x}(\vec{r}) \cdot \int_{S_{p}^{x}} \int_{S_{q}^{x}} \overline{G}^{HE}(\vec{r}, \vec{r}') \cdot f_{q}^{x}(\vec{r}')ds'ds
\]

\[
= - \frac{k_{0}\omega\mu_{m}}{2\pi} \int_{S_{p}^{x}} \int_{S_{q}^{x}} \int_{S_{p}^{x}} \int_{S_{q}^{x}} \left[ f_{p}^{x}(x, y) f_{q}^{x}(x', y') U^{HE} 
- \frac{1}{k_{0}} \frac{\partial f_{p}^{x}(x, y)}{\partial x} \frac{\partial f_{q}^{x}(x', y')}{\partial x'} W^{HE} \right] dx'dy'dxdy
\]

(4.20)
\( Z_{p,q}^{yy} = - \int \int \int \int S_y f_p^y(\vec{r}) \cdot G^{HE}(\vec{r}, \vec{r}') \cdot f_q^y(\vec{r}') ds' ds \)

\[ = - \frac{k_0 \omega \mu_m}{2\pi} \int \int \int \int S_y \int \int \left[ f_p^y(x, y) f_q^y(x', y') U^{HE} \right] ds' ds \]

\[ = - \frac{1}{k_0^2} \frac{\partial f_p^y(x, y)}{\partial y} \frac{\partial f_q^y(x', y')}{\partial x'} W^{HE} \int dx' dy' dx dy, \] (4.21)

and

\( Z_{p,q}^{yx} = - \int \int \int \int S_y f_p^x(\vec{r}) \cdot G^{HE}(\vec{r}, \vec{r}') \cdot f_q^y(\vec{r}') ds' ds \)

\[ = \frac{\omega \mu_m}{2\pi k_0} \int \int \int \int S_y \int \int \left[ f_p^x(x, y) f_q^y(x', y') \right] ds' ds \]

\[ = \frac{1}{k_0^2} \frac{\partial f_p^x(x, y, z)}{\partial z} \frac{\partial f_q^y(x', y', z')}{\partial z'} T^{VE} \int ds' ds, \] (4.22)

Since the testing functions are the same as the expansion functions, clearly \( Z_{q,p}^{yy} = Z_{p,q}^{yx} \), or \( Z^{yx} = (Z^{xy})^T \) and thus the elements in \( Z^{yx} \) can be obtained from \( Z^{yx} \).

The elements in \( Z^{ww} \) can be evaluated in the same way as those in \( Z^{xx} \). However, since the testing and expansion functions can be in different layers and the Green’s function differs depending on the source and testing layers, the appropriate Green’s function must be determined according to the source and testing layers. The elements can be written in the following general form as

\( Z_{p,q}^{ww} = - \int \int \int \int S_w f_p^w(\vec{r}) \cdot G^{VE}(\vec{r}, \vec{r}') \cdot f_q^w(\vec{r}') ds' ds \)

\[ = \frac{k_0 \omega \mu_n}{2\pi} \int \int \int \int S_w \int \int \left[ f_p^w(x, y, z) f_q^w(x', y', z') T^{VE}_a \right] ds' ds \]

\[ = - \frac{1}{k_n^2} \frac{\partial f_p^w(x, y, z)}{\partial z} \frac{\partial f_q^w(x', y', z')}{\partial z'} T^{VE}_b \int ds' ds, \] (4.23)
where \( n \) denotes the layer in which the source, i.e., \( f_w \), locates, and \( T_{a}^{VE} \) and \( T_{b}^{VE} \) are appropriate functions given by

\[
T_{a}^{VE} = \begin{cases} 
V_{1a}^{VE} + V_{1b}^{VE} & \text{both } f_w^p, f_w^q \text{ in 1st layer} \\
V_{na}^{VE} + V_{nb}^{VE} & \text{both } f_w^p, f_w^q \text{ in } n^{th} \text{ layer} \\
V_{ma}^{VE} & f_w^p \text{ in } m^{th} \text{ layer and } f_w^q \text{ in 1st layer} \\
V_{En}^{VE} & f_w^p \text{ in } m^{th} \text{ layer and } f_w^q \text{ in } n^{th} \text{ layer}
\end{cases}
\] (4.24)

and

\[
T_{b}^{VE} = \begin{cases} 
V_{1a}^{VE} - V_{1b}^{VE} & \text{both } f_w^p, f_w^q \text{ in 1st layer} \\
V_{na}^{VE} - V_{nb}^{VE} & \text{both } f_w^p, f_w^q \text{ in } n^{th} \text{ layer} \\
V_{mb}^{VE} & f_w^p \text{ in } m^{th} \text{ layer and } f_w^q \text{ in 1st layer} \\
V_{mb}^{VE} & f_w^p \text{ in } m^{th} \text{ layer and } f_w^q \text{ in } n^{th} \text{ layer}
\end{cases}
\] (4.25)

The same approach can be applied to evaluate elements in \( Z_{xw} \) and \( Z_{yw} \), which can be given by

\[
Z_{xw}^{p,q} = - \int \int_{S_p} f_p^x(\bar{r}) \cdot \int \int_{S_q} G^{VE}(\bar{r}, \bar{r}') \cdot f_q^w(\bar{r}') ds'ds
\]

\[
= - \frac{\omega \mu_0 \epsilon_0}{2\pi \epsilon_m k_0} \int \int_{S_p} \int \int_{S_q} \frac{\partial f_p^x(x, y) \partial f_q^w(x', y', z')}{\partial x} W_{VEn} ds'dx dy,
\] (4.26)

\[
Z_{yw}^{p,q} = - \int \int_{S_p} f_p^y(\bar{r}) \cdot \int \int_{S_q} G^{VE}(\bar{r}, \bar{r}') \cdot f_q^w(\bar{r}') ds'ds
\]

\[
= - \frac{\omega \mu_0 \epsilon_0}{2\pi \epsilon_m k_0} \int \int_{S_p} \int \int_{S_q} \frac{\partial f_p^y(x, y) \partial f_q^w(x', y', z')}{\partial y} W_{VEn} ds'dx dy,
\] (4.27)

where \( n \) denotes the layer in which \( f_w^q \) locates. Once \( Z_{xw} \) and \( Z_{yw} \) are computed, \( Z_{wx} \) and \( Z_{wy} \) can be obtained from \( Z_{wx} = (Z_{xw})^T \) and \( Z_{wy} = (Z_{yw})^T \).

Finally, to evaluate the elements in submatrices involving the attachment mode basis functions, a special care is required since the functions have two parts, namely, wire part
and patch part. The elements in $Z^{aa}$ can be given as

$$
Z_{p,q}^{aa} = \int\int_{S_p^w} f_p^{aw}(\bar{r}) \cdot \int\int_{S_q^w} \mathcal{C}^{VE}(\bar{r}, \bar{r}') \cdot f_q^{aw}(\bar{r}') ds' ds
+ \int\int_{S_p^w} f_p^{ap}(\bar{r}) \cdot \int\int_{S_q^p} \mathcal{C}^{HE}(\bar{r}, \bar{r}') \cdot f_q^{ap}(\bar{r}') ds' ds
+ \int\int_{S_p^w} f_p^{aw}(\bar{r}) \cdot \int\int_{S_q^p} \mathcal{C}^{HE}(\bar{r}, \bar{r}') \cdot f_q^{ap}(\bar{r}') ds' ds
+ \int\int_{S_p^p} f_p^{ap}(\bar{r}) \cdot \int\int_{S_q^p} \mathcal{C}^{HE}(\bar{r}, \bar{r}') \cdot f_q^{ap}(\bar{r}') ds' ds
= I_1 + I_2 + I_3 + I_4. \tag{4.28}
$$

The first term in the right hand side, $I_1$, represents the coupling between two wire mode segments and thus can be evaluated in the same way as elements in $Z^{aw}$ given by (4.23). Likewise, $I_4$ represents the coupling between two patch part attachment mode, which can be evaluated by

$$
I_4 = - \frac{k_0 \omega \mu_m}{2\pi} \int\int_{S_p^p} \int\int_{S_q^p} \left[ f_p^{ap}(x, y) f_q^{ap}(x', y') U^{HE} \cos(\phi - \phi') - \frac{1}{k_0^2} W^{HE} \frac{\partial}{\partial \rho} \rho f_p^{ap}(x, y) \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho f_q^{ap}(x', y') \right] ds' ds. \tag{4.29}
$$

$I_2$ is the coupling between the patch part attachment mode and the electric field due to the wire part attachment mode, which can be given by

$$
I_2 = - \frac{\omega \mu_0 \epsilon_0}{2\pi \epsilon_m k_0} \int\int_{S_p^w} \int\int_{S_q^w} W^{VE} \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho f_p^{ap}(x, y) \frac{\partial f_q^{aw}(x', y', z')}{\partial z'} ds' ds, \tag{4.30}
$$

where the wire part is assumed to be in the $n^{th}$ layer. Using the reciprocity theorem, $I_3$ can be in the same way as $I_2$ as

$$
I_3 = - \frac{\omega \mu_0 \epsilon_0}{2\pi \epsilon_m k_0} \int\int_{S_q^p} \int\int_{S_p^w} W^{VE} \frac{1}{\rho'} \frac{\partial}{\partial \rho'} \rho' f_q^{ap}(x', y') \frac{\partial f_p^{aw}(x, y, z)}{\partial z} ds' ds. \tag{4.31}
$$
Next, the elements in submatrix $Z_{aw}$ are given by

\[
Z_{aw}^{pq} = \int \int_{S_{p}^{aw}} f_{p}^{aw}(\vec{r}) \cdot \int \int_{S_{q}^{w}} C^{VE} (\vec{r}, \vec{r}') \cdot f_{q}^{w}(\vec{r}') ds' ds \\
+ \int \int_{S_{p}^{ap}} f_{p}^{ap}(\vec{r}) \cdot \int \int_{S_{q}^{w}} C^{HE} (\vec{r}, \vec{r}') \cdot f_{q}^{w}(\vec{r}') ds' ds,
\]

which can be computed in the same way as $I_1$ and $I_2$ of $Z_{aa}^{p,q}$, and $Z_{wa}$ can be obtained from $Z_{wa} = (Z_{aw})^T$. Finally, the elements in $Z_{xa}$ and $Z_{ya}$ can be computed by

\[
Z_{xa}^{pq} = \int \int_{S_{p}^{x}} f_{p}^{x}(\vec{r}) \cdot \int \int_{S_{q}^{aw}} C^{VE} (\vec{r}, \vec{r}') \cdot f_{q}^{aw}(\vec{r}') ds' ds \\
+ \int \int_{S_{p}^{ap}} f_{p}^{ap}(\vec{r}) \cdot \int \int_{S_{q}^{w}} C^{HE} (\vec{r}, \vec{r}') \cdot f_{q}^{w}(\vec{r}') ds' ds \\
= -\frac{\omega \mu_{0} \epsilon_{0}}{2\pi \epsilon_{m} k_{0}} \left[ \int \int_{S_{p}^{x}} \int \int_{S_{q}^{aw}} \frac{\partial f_{p}^{x}(x, y)}{\partial x} \frac{\partial f_{q}^{aw}(x', y', z')}{\partial z'} W^{VE}_{En} ds' dx dy \right. \\
+ \left. \int \int_{S_{p}^{ap}} \int \int_{S_{q}^{w}} W^{HE} \frac{\partial f_{p}^{x}(x, y)}{\partial x} \frac{1}{\rho'} \frac{\partial}{\partial \rho'} f_{q}^{ap}(x', y') ds' dx dy \right], \tag{4.33}
\]

and

\[
Z_{ya}^{pq} = \int \int_{S_{p}^{y}} f_{p}^{y}(\vec{r}) \cdot \int \int_{S_{q}^{aw}} C^{VE} (\vec{r}, \vec{r}') \cdot f_{q}^{aw}(\vec{r}') ds' ds \\
+ \int \int_{S_{p}^{ap}} f_{p}^{ap}(\vec{r}) \cdot \int \int_{S_{q}^{w}} C^{HE} (\vec{r}, \vec{r}') \cdot f_{q}^{w}(\vec{r}') ds' ds \\
= -\frac{\omega \mu_{0} \epsilon_{0}}{2\pi \epsilon_{m} k_{0}} \left[ \int \int_{S_{p}^{y}} \int \int_{S_{q}^{aw}} \frac{\partial f_{p}^{y}(x, y)}{\partial y} \frac{\partial f_{q}^{aw}(x', y', z')}{\partial z'} W^{VE}_{En} ds' dx dy \right. \\
+ \left. \int \int_{S_{p}^{ap}} \int \int_{S_{q}^{w}} W^{HE} \frac{\partial f_{p}^{y}(x, y)}{\partial y} \frac{1}{\rho'} \frac{\partial}{\partial \rho'} f_{q}^{ap}(x', y') ds' dx dy \right], \tag{4.34}
\]
where the wire part is assumed to be in the $n^{th}$ layer as before. The submatrices $Z^{ax}$ and $Z^{ay}$ can be found from $Z^{ax} = (Z^{xa})^T$, and $Z^{ay} = (Z^{ya})^T$, respectively. This will complete the computation process of the matrix $Z$ in (4.16).

The right hand side vector $\mathbf{v}$ in (4.16) can be evaluated in the same way as $Z$ but computing it requires only one integral on the testing function. Its elements can be given by

$$v^x_p = \int \int_{S^x_p} \mathbf{f}^x_p(\vec{r}) \cdot \mathbf{E}^{inc} ds,$$  
(4.35)

$$v^y_p = \int \int_{S^y_p} \mathbf{f}^y_p(\vec{r}) \cdot \mathbf{E}^{inc} ds,$$  
(4.36)

$$v^w_p = \int \int_{S^w_p} \mathbf{f}^w_p(\vec{r}) \cdot \mathbf{E}^{inc} ds,$$  
(4.37)

and

$$v^a_p = \int \int_{S^{aw}_p} \mathbf{f}^{aw}_p(\vec{r}) \cdot \mathbf{E}^{inc} ds + \int \int_{S^{ap}_p} \mathbf{f}^{ap}_p(\vec{r}) \cdot \mathbf{E}^{inc} ds.$$  
(4.38)

It is noted that $\mathbf{E}^{inc}$ depends on the excitation methods as discussed in section 4.1.

Once $Z$ and $\mathbf{v}$ are computed, the MoM matrix equation can be solved to obtain $\mathbf{i}$, which contains the unknown coefficients associated with expansion functions. The fast matrix-vector multiplication using the fast Fourier transform (FFT) and the DFT-based preconditioner, which can be used in iterative solvers implemented in this work, will be discussed in the next chapter. This latter preconditioned iterative MoM method which is developed below is referred to as the PI-MoM method as indicated previously in chapter 1.
CHAPTER 5

FAST MATRIX-VECTOR MULTIPLICATION AND PRECONDITIONER FOR THE PI-MOM APPROACH

In the previous chapter, the integral equation governing the radiation/scattering problem of finite arrays is formulated and the method of moments is applied to obtain the MoM matrix equation. In this chapter, the MoM matrix equation will be solved to obtain the unknown coefficient vector, which together with the known expansion functions represent the unknown currents. To solve the matrix equation, a direct solver can be applied; however it has two great disadvantages, namely,

1. it has to store the entire matrix, thus it requires large amount of memory storage,

2. its computational cost is $O(N^3)$ for $N$ unknowns, which is considered extremely expensive.

These disadvantages lead to the limitation in the size of arrays since the number of unknowns is proportional to the number of array elements.

Therefore, it is desirable to implement a solver which can reduce not only the storage requirement but also the computational cost. It is well-known that the MoM operator matrix for the periodic array has the Toeplitz property due to the the shift-invariant nature of the
Green’s functions, thus only few rows and columns are needed, leading to significant reduction in memory storage. Also, an iterative solver is found to have cheaper computational cost, if the solution converges after only few iterations. Based on these two approaches, an effective iterative solver may be developed to both reduce the storage and improve the efficiency.

Although an iterative solver can save solving time, its convergence, which depends on the condition number of the matrix, is not guaranteed. Since large matrices can generally have a large condition number, poor convergence is expected for very large array problems, and thus a preconditioner is typically required to make the iterative solver practical. In this work, an efficient DFT-based preconditioner is developed to be used with several well-known iterative solvers, such as conjugate-gradient, biconjugate-gradient stabilized and so on. Such a preconditioned iterative MoM (PI-MoM) technique is developed in this chapter.

This chapter consists of four major parts. In section 5.1, the general structure and property of the MoM operator matrix for finite periodic arrays are discussed. By exploiting its property, an efficient matrix-vector multiplication method using FFT can be applied and it will be described in section 5.2. Section 5.3 discusses the DFT-based preconditioner developed in this work leading to the PI-MoM approach, and some numerical results based on this PI-MoM are presented in section 5.4.

5.1 Block-Toeplitz Property of the MoM Operator Matrix

When the finite periodic array of interest is one-dimensional, i.e., it has only one row or one column, if only one expansion mode is used to represent the current on each array element, then due to the property of the Green’s function, which is a function of only $|\vec{r} - \vec{r'}|$, then it follows that $Z_{2,1} = Z_{3,2} = Z_{4,3}, Z_{3,1} = Z_{4,2} = Z_{5,3}$, and vice versa. Therefore, the
MoM operator matrix will be a $M \times M$ Toeplitz matrix, which can be written as follows

$$
T_M = \begin{bmatrix}
t_0 & t_{-1} & t_{-2} & \cdots & t_{-M+1} \\
t_1 & t_0 & t_{-1} & \cdots & t_{-M+2} \\
t_2 & t_1 & t_0 & \cdots & t_{-M+3} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
t_{M-1} & t_{M-2} & t_{M-3} & \cdots & t_0
\end{bmatrix},
$$

(5.1)

where $M$ is the dimension of the matrix. Clearly, for general Toeplitz matrices, knowledge of only one row and one column is sufficient to determine the whole matrix. Furthermore, since the MoM operator matrix is symmetric, it is clear that $t_m = t_{-m}$ and thus only one row or column is sufficient to represent the whole matrix.

Now, for a two-dimensional $M \times N$ finite periodic array using one expansion mode per element, the MoM operator matrix can be written as follows

$$
Z = \begin{bmatrix}
Z_{1,1} & Z_{1,2} & Z_{1,3} & \cdots & Z_{1,N} \\
Z_{2,1} & Z_{2,2} & Z_{2,3} & \cdots & Z_{2,N} \\
Z_{3,1} & Z_{3,2} & Z_{3,3} & \cdots & Z_{3,N} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
Z_{N,1} & Z_{N,2} & Z_{N,3} & \cdots & Z_{N,N}
\end{bmatrix},
$$

(5.2)

where each submatrix $Z_{i,j}$ represents the couplings among elements in the $i^{th}$ and $j^{th}$ rows and is given by

$$
Z_{i,j} = \begin{bmatrix}
(Z_{i,j})_{1,1} & (Z_{i,j})_{1,2} & (Z_{i,j})_{1,3} & \cdots & (Z_{i,j})_{1,M} \\
(Z_{i,j})_{2,1} & (Z_{i,j})_{2,2} & (Z_{i,j})_{2,3} & \cdots & (Z_{i,j})_{2,M} \\
(Z_{i,j})_{3,1} & (Z_{i,j})_{3,2} & (Z_{i,j})_{3,3} & \cdots & (Z_{i,j})_{3,M} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
(Z_{i,j})_{M,1} & (Z_{i,j})_{M,2} & (Z_{i,j})_{M,3} & \cdots & (Z_{i,j})_{M,M}
\end{bmatrix},
$$

(5.3)

with $(Z_{i,j})_{k,l}$ representing the coupling between the $k^{th}$ element in the $i^{th}$ row and the $l^{th}$ element in the $j^{th}$ row or $(k,i)^{th}$ and $(l,j)^{th}$ elements. The matrix in this case will become a block-Toeplitz matrix with Toeplitz blocks (BTTB), which can be given by

$$
T_{NM} = \begin{bmatrix}
T_0 & T_{-1} & T_{-2} & \cdots & T_{-N+1} \\
T_1 & T_0 & T_{-1} & \cdots & T_{-N+2} \\
T_2 & T_1 & T_0 & \cdots & T_{-N+3} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
T_{N-1} & T_{N-2} & T_{N-3} & \cdots & T_0
\end{bmatrix},
$$

(5.4)

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where each submatrix $T_k$ is a symmetric Toeplitz matrix given in (5.1). Here, the matrix $T_{NM}$ given in (5.4) is said to be a $NM \times NM$ BTTB matrix, which consists of $N \times N$ blocks with each block being a $M \times M$ matrix. It can be straightforwardly shown that $T_k$ is symmetric and $T_k = T_{-k}$. Thus only one row or column is again sufficient to represent the whole matrix.

Following the same approach, it can be extended to problems where multiple expansion modes are used to expand the current on each array element on two-dimensional arrays. Assume that $N_{\text{mode}}$ expansion modes per element are used in analyzing a $M \times N$ two-dimensional array, the MoM operator matrix in this case can be written as follows

$$
Z = \begin{bmatrix}
Z_{1,1} & Z_{1,2} & Z_{1,3} & \ldots & Z_{1,N_{\text{mode}}} \\
Z_{2,1} & Z_{2,2} & Z_{2,3} & \ldots & Z_{2,N_{\text{mode}}} \\
Z_{3,1} & Z_{3,2} & Z_{3,3} & \ldots & Z_{3,N_{\text{mode}}} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
Z_{N_{\text{mode}},1} & Z_{N_{\text{mode}},2} & Z_{N_{\text{mode}},3} & \ldots & Z_{N_{\text{mode}},N_{\text{mode}}}
\end{bmatrix},
$$

(5.5)

where the submatrix $Z^{k,l}$ represents the couplings between the $k^{th}$ and $l^{th}$ modes and it is found to be a $NM \times NM$ BTTB matrix. It is noted that each submatrix has the same structure as that given in (5.4), and Toeplitz blocks in each BTTB submatrix may not be symmetric. Since $Z$ is symmetric, it is evident that $Z^{k,l} = (Z^{l,k})^T$. Although $Z^{k,l}$ is BTTB, the whole matrix $Z$ itself is not, i.e., $Z^{k,l}$ is in general not equal to $Z^{k+1,l+1}$ as in a BTTB matrix, thus for the worst case it can require up to $\frac{N_{\text{mode}}^2 + N_{\text{mode}}}{2}$ BTTB submatrices to represent the whole matrix with $2MN$ storage for each submatrix. However, it is found that in most cases the storage requirement can be significantly reduced by utilizing the symmetry of the expansion functions. In conclusion, by exploiting the periodicity of the array, the storage can drastically be reduced to $O(N_{\text{total}})$ compared to $O(N_{\text{total}}^2)$ required for the full matrix.
Table 5.1: Number of expansion modes per array element

<table>
<thead>
<tr>
<th>Element Type</th>
<th>$N_{\text{mode}}$</th>
<th>$N_{\text{mode},x}$</th>
<th>$N_{\text{mode},y}$</th>
<th>$N_{\text{mode},w}$</th>
<th>$N_{\text{mode},a}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dipole</td>
<td>$N_{\text{mode}}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Microstrip-line fed patch array: radiation</td>
<td>$N_{\text{mode},x}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Microstrip-line fed patch array: scattering</td>
<td>$N_{\text{mode},x} + N_{\text{mode},y}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Probe fed patch array</td>
<td>$N_{\text{mode},x} + N_{\text{mode},y} + N_{\text{mode},w} + N_{\text{mode},a}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Aperture-coupled patch array</td>
<td>$N_{\text{mode},x} + N_{\text{mode},y}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

To apply the approach mentioned above to the $Z$ matrix in (4.17) simply requires the rearrangement of the matrix by expansion modes in order to obtain the structure given in (5.5). Let $N_{\text{mode},x}$, $N_{\text{mode},y}$, $N_{\text{mode},w}$, and $N_{\text{mode},a}$ be the number of $\hat{x}$-directed patch modes, $\hat{y}$-directed patch modes, wire modes, and attachment modes, per array element, respectively, then $N_{\text{mode}}$ for each array element is given in table (5.1). Clearly, $Z$ consists of $N_{\text{mode}}^2$ BTTB submatrices of dimension $NM \times NM$ and the total number of unknowns for a $M \times N$ array is $N_{\text{mode}} \times N \times M$.

5.2 Fast Matrix-Vector Multiplication Using FFT

A good iterative solver requires only few iterations for the solution to converge, thus the computational cost is roughly proportional to the computational cost of the matrix-vector multiplications required in each iteration. Therefore, the efficiency of an iterative solver can be improved by accelerating the matrix-vector multiplication process, which typically requires $O(N)$ operations for a $N \times N$ matrix. For structured matrices like a Toeplitz matrix, or a BTTB matrix, there exist fast matrix-vector multiplication methods which
can significantly reduce the computational cost. This section will describe a fast matrix-vector multiplication method using the fast Fourier transform (FFT) which can be applied to BTTB matrices.

A $M \times M$ circulant matrix is a special $M \times M$ Toeplitz matrix given by

$$C_M = \begin{bmatrix}
c_0 & c_{M-1} & c_{M-2} & \cdots & c_1 \\
c_1 & c_0 & c_{M-1} & \cdots & c_2 \\
c_2 & c_1 & c_0 & \cdots & c_3 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
c_{M-1} & c_{M-2} & c_{M-3} & \cdots & c_0
\end{bmatrix}. \quad (5.6)$$

Compared to a Toeplitz matrix in (5.1), it is clear that a circulant matrix is the Toeplitz matrix which has $t_{-k} = t_{M-k}$. Circulant matrices are diagonalized by the FFT, i.e.,

$$C_M = F_M^H \Lambda_M F_M, \quad (5.7)$$

where

$$\Lambda_M = \text{diag}(F_M \mathbf{c}_0) = \begin{bmatrix}
(F_M \mathbf{c}_0)_1 & 0 & 0 & \cdots & 0 \\
0 & (F_M \mathbf{c}_0)_2 & 0 & \cdots & 0 \\
0 & 0 & (F_M \mathbf{c}_0)_3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & (F_M \mathbf{c}_0)_M
\end{bmatrix}. \quad (5.8)$$

$F_M$ is the $M \times M$ Fourier matrix whose elements are given by $(F_M)_{k,l} = \frac{1}{\sqrt{M}} e^{-2\pi j(k-1)(l-1)/M}$, and $\mathbf{c}_0$ is the first column of $C$, i.e., $\mathbf{c}_0 = [c_0, c_1, \cdots c_{M-1}]^T$. Also, $F_M^H$ denotes the Hermitian of $F_M$. It is well-known that $F_M$ is a unitary matrix, i.e., $F_M^H F_M = F_M F_M^H = \hat{I}_M$, where $\hat{I}_M$ denotes the $M \times M$ identity matrix. The diag($\bullet$) operator introduced here is defined as

$$\begin{align}
\text{diag}(\mathbf{c}) &= \Lambda, \quad \text{where } \Lambda \text{ is the diagonal matrix whose main diagonal is the vector } \mathbf{c} \\
\text{diag}(A) &= \mathbf{x}, \quad \text{where } \mathbf{x} \text{ is the vector that contains the diagonal elements of } A.
\end{align} \quad (5.9)$$
Let $\mathbf{x}$ be a vector of length $M$, then the matrix-vector multiplication $C_M \mathbf{x}$ can then be computed by

$$
C_M \mathbf{x} = F_M^H \mathbf{A}_M F_M \mathbf{x} \\
= F_M^H \text{diag}(F_M \mathbf{c}_0) \mathcal{F}[\mathbf{x}] \\
= \mathcal{F}^{-1} \{ \mathcal{F}[\mathbf{c}_1] \ast \mathcal{F}[\mathbf{x}] \},
$$

(5.10)

where $\mathcal{F}[\mathbf{x}]$ denotes the discrete Fourier transform of the vector $\mathbf{x}$ and only here does the operator $\ast$ denote the element-wise multiplication and not the complex conjugate operator, i.e., $(\mathbf{x} \ast \mathbf{y})_k = x_k y_k$. Since the discrete Fourier transform can be performed by using FFT, which is a $\mathcal{O}(N \log N)$ operation, the computation cost of this matrix-vector multiplication is $\mathcal{O}(M \log M)$.

To apply the approach mentioned above to a Toeplitz matrix given in (5.1), the Toeplitz matrix of interest is first embedded into a $2M \times 2M$ circulant matrix as

$$
C_{2M} = \begin{bmatrix}
T_M & B_M \\
B_M & T_M
\end{bmatrix},
$$

(5.11)

where

$$
B_M = \begin{bmatrix}
0 & t_{M-1} & t_{M-2} & \cdots & t_1 \\
t_{-M+1} & 0 & t_{M-1} & \cdots & t_2 \\
t_{-M+2} & t_{-M+1} & 0 & \cdots & t_3 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
t_{-1} & t_{-2} & t_{-3} & \cdots & 0
\end{bmatrix}.
$$

(5.12)

Let $\mathbf{0}$ be the zero vector of length $M$ and $\mathbf{y}^T = [\mathbf{x}^T \mathbf{0}^T]$, then the product $T_M \mathbf{x}$ can then be computed from

$$
C_{2M} \mathbf{y} = \begin{bmatrix}
T_M & B_M \\
B_M & T_M
\end{bmatrix} \begin{bmatrix}
\mathbf{x} \\
\mathbf{0}
\end{bmatrix} = \begin{bmatrix}
T_M \mathbf{x} \\
B_M \mathbf{x}
\end{bmatrix}.
$$

(5.13)

Since $C_{2M}$ is a circulant matrix, $C_{2M} \mathbf{y}$ can be computed using FFT with $\mathcal{O}(M \log M)$ operations. Therefore, the multiplication $T_M \mathbf{x}$ is also a $\mathcal{O}(M \log M)$ operation.
Likewise, a $NM \times NM$ block circulant matrix with circulant blocks (BCCB) is a special $NM \times NM$ BTTB matrix given by

$$C_{NM} = \begin{bmatrix} C_0 & C_{N-1} & C_{N-2} & \cdots & C_1 \\ C_1 & C_0 & C_{N-1} & \cdots & C_2 \\ C_2 & C_1 & C_0 & \cdots & C_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_{N-1} & C_{N-2} & C_{N-3} & \cdots & C_0 \end{bmatrix},$$

(5.14)

where each submatrix $C_k$ is a $M \times M$ circulant matrix given in (5.6). BCCB matrices can also be diagonalized by the FFT, i.e.,

$$C_{NM} = F_{NM}^H \Lambda_{NM} F_{NM}$$

$$= F_{NM}^H \text{diag}(F_{NM} c_0) F_{NM},$$

(5.15)

where

$$F_{NM} = F_N \otimes F_M,$$

(5.16)

and the operator $\otimes$ denotes the Kronecker product. Also, $c_0$ is the first column of $C_{NM}$ as before. It is worthwhile noting that $F_{NM}$ corresponds to the two-dimensional discrete Fourier transform. Let $\mathbf{x}$ be a vector of length $NM$, then the product $C_{NM} \mathbf{x}$ can then be computed by

$$C_{NM} \mathbf{x} = F_{NM}^H \Lambda_{NM} F_{NM} \mathbf{x}$$

$$= F_{NM}^H \text{diag}(F_{NM} c_0) \mathbf{F}_2[\mathbf{x}]$$

$$= \mathbf{F}_2^{-1} \{ \mathbf{F}_2[c_0] \ast \mathbf{F}_2[\mathbf{x}] \},$$

(5.17)

where $\mathbf{F}_2[\mathbf{x}]$ denotes the two-dimensional discrete Fourier transform of the vector $\mathbf{x}$. The computation cost of this matrix-vector multiplication is found to be $O(NM \log NM)$.

The approach mentioned above can be easily applied to a $NM \times NM$ BTTB matrix, $T_{NM}$ by embedding it into a $4NM \times 4NM$ BCCB matrix. To construct the BCCB matrix,
each Toeplitz block is first embedded into a circulant matrix to obtain
\[
(C_{2M})_k = \begin{bmatrix}
    (T_M)_k & (B_M)_k \\
    (B_M)_k & (T_M)_k
\end{bmatrix},
\] (5.18)
where \((T_M)_k\) is the \(k^{th}\) \(M \times M\) Toeplitz submatrix, and \((B_M)_k\) can be constructed from \((T_M)_k\) using (5.12). Next, embedding all circulant submatrices into a BCCB matrix by
\[
C_{4NM} = \begin{bmatrix}
    T_{2NM} & B_{2NM} \\
    B_{2NM} & T_{2NM}
\end{bmatrix},
\] (5.19)
where
\[
T_{2NM} = \begin{bmatrix}
    (C_{2M})_0 & (C_{2M})_{-1} & (C_{2M})_{-2} & \cdots & (C_{2M})_{-N+1} \\
    (C_{2M})_1 & (C_{2M})_0 & (C_{2M})_{-1} & \cdots & (C_{2M})_{-N+2} \\
    (C_{2M})_2 & (C_{2M})_1 & (C_{2M})_0 & \cdots & (C_{2M})_{-N+3} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    (C_{2M})_{N-1} & (C_{2M})_{N-2} & (C_{2M})_{N-3} & \cdots & (C_{2M})_0
\end{bmatrix},
\] (5.20)
and
\[
B_{2NM} = \begin{bmatrix}
    0 & (C_{2M})_{M-1} & (C_{2M})_{M-2} & \cdots & (C_{2M})_1 \\
    (C_{2M})_{M+1} & 0 & (C_{2M})_{M-1} & \cdots & (C_{2M})_2 \\
    (C_{2M})_{M+2} & (C_{2M})_{M+1} & 0 & \cdots & (C_{2M})_3 \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    (C_{2M})_{-1} & (C_{2M})_{-2} & (C_{2M})_{-3} & \cdots & 0
\end{bmatrix},
\] (5.21)
where 0 denotes the \(2M \times 2M\) zero matrix. Note that both \(T_{2NM}\) and \(B_{2NM}\) are block Toeplitz matrices with circulant blocks (BTCB). Now, consider a vector \(\mathbf{x}_{NM}\) of length \(NM\) given by \(\mathbf{x}_{NM} = [\mathbf{x}_1^T \mathbf{x}_2^T \cdots \mathbf{x}_N^T]^T\), where \(\mathbf{x}_k, k = 1, \ldots, N\) is a vector of length \(M\). Let \(\mathbf{y}_{4NM}\) be a vector constructed from \(\mathbf{x}_{NM}\) by
\[
\mathbf{y}_{4NM} = \begin{bmatrix}
    \mathbf{x}_1 \\
    \mathbf{0} \\
    \mathbf{x}_2 \\
    \mathbf{0} \\
    \vdots \\
    \mathbf{x}_N \\
    \mathbf{0} \\
    \mathbf{0}_{2NM}
\end{bmatrix},
\] (5.22)
where $\mathbf{0}$ is the zero vector of length $M$ as before and $\mathbf{0}_{2NM}$ is the zero vector of length $2NM$. The product $T_{NM}\mathbf{x}_{NM}$ can then be computed from

$$C_{4NM}y_{4NM} = \begin{bmatrix} T_{2NM} & B_{2NM} \\ B_{2NM} & T_{2NM} \end{bmatrix} \begin{bmatrix} y_{2NM} \\ 0_{2NM} \end{bmatrix} = \begin{bmatrix} T_{2NM}y_{2NM} \\ B_{2NM}y_{2NM} \end{bmatrix}.$$ 

(5.23)

Since $C_{4NM}$ is a BCCB matrix, $C_{4NM}y_{4NM}$ can be computed using FFT, and thus the multiplication $T_{NM}\mathbf{x}_{NM}$ can be obtained in $O(NM \log NM)$ operations.

As mentioned in the previous section, the MoM operator matrix can be arranged such that each submatrix is a BCCB matrix, and thus the multiplication of each submatrix and a vector can be performed using FFT to reduce the computational cost. It is also noted that the embedding BTTB matrix into BCCB matrix and the Fourier transform of the BCCB matrix can be precomputed and stored in order to avoid redundant computations and improve the efficiency.

### 5.3 A DFT-based Preconditioner and the PI-MoM Development

As mentioned earlier, the matrix equation given in (4.16) can be solved by an iterative solver to avoid storing the whole matrix required by the direct solver. However, it is known that the convergence rate of the iterative solvers depends on the spectral properties and the condition number of MoM operator matrix[49]. The convergence rate can be improved by employing an appropriate preconditioner for a particular system, which results in the following matrix equation

$$M^{-1}Z\hat{\mathbf{z}} = M^{-1}\mathbf{y},$$

(5.24)

where $M^{-1}$ denotes a preconditioner. The matrix $M$ is chosen such that it well approximates $Z$ to improve the condition number of the preconditioned system given above. Furthermore, it is also desirable that computing $M$ as well as solving $M\mathbf{x} = \mathbf{y}$ should not incur large computational cost in order to achieve good efficiency.
In this work, the preconditioner is constructed in the DFT domain by approximating the transformed impedance matrix using only diagonal elements of each block, then the preconditioner in the original domain can be simply obtained by the inverse transform of the block-diagonal approximated matrix. Consider first a $M \times N$ periodic array problem using only one expansion mode per array element. To construct the DFT-based preconditioned system, one may first apply the two-dimensional discrete Fourier transform (DFT) to the matrix equation in (4.16) to yield

$$ F_{NM} Z F_{NM}^H F_{NM} \tilde{z} = F_{NM} \tilde{v}, $$

where $F_{NM}$ denotes the two-dimensional DFT matrix given previously in (5.16). Introducing identities $\tilde{Z} = F_{NM} Z F_{NM}^H$, $\tilde{z} = F_{NM} \tilde{z}$, and $\tilde{v} = F_{NM} \tilde{v}$, respectively into (5.25) allows one to rewrite (5.25) as

$$ \tilde{Z} \tilde{z} = \tilde{v}. $$

It can be shown by inspection that the matrix $\tilde{Z}$ is highly sparse and nearly diagonal, because the field due to a DFT basis function has strong interactions with only few close-by complex conjugate DFT testing functions. Let $\tilde{Z}_d$ be the diagonal approximated matrix of $\tilde{Z}$, i.e.,

$$ \tilde{Z}_d = \text{diag}(\tilde{z}_d), $$

where $\tilde{z}_d$ is the vector that contains all diagonal elements of $\tilde{Z}$ given by

$$ \tilde{z}_d = \text{diag}(\tilde{Z}) = [\tilde{Z}_{1,1}, \tilde{Z}_{2,2}, \ldots, \tilde{Z}_{NM,NM}]^T, $$

and employing $\tilde{Z}_d^{-1}$ as the preconditioner in the transform domain, i.e.,

$$ \tilde{M}^{-1} = \tilde{Z}_d^{-1}, $$

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allows one to obtain the preconditioner in the original domain by its inverse transform as

\[ M^{-1} = F_{NM}^H \tilde{M}^{-1} F_{NM} = F_{NM}^H \tilde{Z}_d^{-1} F_{NM} \]  

(5.30)

and the preconditioned system in (5.24) can be rewritten as

\[ (F_{NM}^H \tilde{Z}_d^{-1} F_{NM}) \tilde{z} = (F_{NM}^H \tilde{Z}_d^{-1} F_{NM}) v. \]  

(5.31)

From (5.30), it is evident that the matrix equation \( M \bar{x} = \bar{y} \) can be solved by

\[
\begin{align*}
\bar{x} &\equiv M^{-1} \bar{y} \\
&= F_{NM}^H \tilde{Z}_d^{-1} F_{NM} \bar{y} \\
&= F_{NM}^H \tilde{z}_d^{-1} \mathcal{F}_2 [\bar{y}] \\
&= F_{NM}^H (\tilde{z}_d^{-1} \cdot \mathcal{F}_2 [\bar{y}]) \\
&= \mathcal{F}_2^{-1} \{ \tilde{z}_d^{-1} \cdot \mathcal{F}_2 [\bar{y}] \},
\end{align*}
\]

(5.32)

where \( \cdot \) denotes the element-wise multiplication as before and \( \tilde{z}_d^{-1} = [\tilde{z}_1^{-1} \tilde{z}_2^{-1} \cdots \tilde{z}_{NM,NM}^{-1}]^T \).

It is worthwhile noting that the DFT operations needed in the above system can be efficiently performed using the fast Fourier transform algorithm resulting in a fast iterative solver. Also, one notes that one can utilize \( \tilde{M}^{-1} = \tilde{Z}^{-1} \) instead of the approximation to \( \tilde{Z}^{-1} \) as in (5.29); however, the use of \( \tilde{M}^{-1} = \tilde{Z}^{-1} \) would increase the computational time. Alternatively, one may only retain the most significant terms in \( \tilde{Z}^{-1} \), instead of only the diagonal elements \( \tilde{Z}_d^{-1} \), such that the terms retained are above a predetermined threshold in order to improve the performance of the preconditioner.

It is evident from (5.15) that the preconditioner \( M \) constructed here from (5.29) is a BCCB matrix. From the mathematical viewpoint, it is called the BCCB preconditioner. Previous studies show that this BCCB preconditioner is a good approximation to a BTTB
It can be mathematically defined as the minimizer of \( \|C_{NM} - T_{NM}\|_F \) over all \( C_{NM} \in \mathcal{C}^{NM \times NM} \), i.e.,

\[
M = \arg \min_{C_{NM} \in \mathcal{C}^{NM \times NM}} \|C_{NM} - T_{NM}\|_F, \tag{5.33}
\]

where \( \mathcal{C}^{NM \times NM} \) denotes the space of all \( NM \times NM \) BCCB matrices, \( T_{NM} \) is the approximated \( NM \times NM \) BTTB matrix, and \( \|A\|_F \) denotes the Frobenius norm of \( A \) defined as

\[
\|A\|_F = \sqrt{\sum_{i=1}^{N} \sum_{j=1}^{M} |a_{ij}|^2} \tag{5.34}
\]

for a \( N \times M \) matrix. In fact, it is the closest BCCB matrix to the BTTB matrix in terms of Frobenius norm, since

\[
\|C_{NM} - T_{NM}\|_F = \|F_{NM}^H (C_{NM} - T_{NM}) F_{NM}\|_F
= \|A_{NM} - F_{NM}^H T_{NM} F_{NM}\|_F. \tag{5.35}
\]

Therefore, choosing \( \text{diag}(A_{NM}) = \text{diag}(F_{NM}^H T_{NM} F_{NM}) \) will minimize the Frobenius norm. This is exactly what is given in (5.27).

To calculate the diagonal elements of \( \tilde{Z} \), first notice that

\[
\tilde{Z}_{i,i} = \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} \sum_{n'=0}^{N-1} \sum_{m'=0}^{M-1} z_{nm,n'm'} e^{-j2\pi(p-n')p/N} e^{-j2\pi(n-n')q/M}. \tag{5.36}
\]

The index \( i \) \((i = 1, \cdots, NM)\) corresponds to the double index \((p, q)\) of the DFT terms, where \( p = 0, \cdots, N - 1 \) and \( q = 0, \cdots, M - 1 \). The element double indices \( nm \) and \( n'm' \) denote the testing and expansion modes, respectively, i.e., \( z_{nm,n'm'} \) represents the coupling between the \((m+1)^{th}\) element in the \((n+1)^{th}\) row and \((m'+1)^{th}\) element in the \((n'+1)^{th}\) row. As mentioned earlier, the MoM operator matrix is a BTTB matrix, and \( z_{nm,n'm'} \) is a
function of only \((n - n', m - m')\). Hence, applying the variable transformations \(k = n - n'\) and \(l = m - m'\) to (5.36) yields

\[
\tilde{Z}_{i,i} = \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} \left\{ \alpha_{k,l} z_{k,l} e^{-j2\pi kp/N} e^{-j2\pi lq/M} + \alpha_{-k,l} z_{-k,l} e^{j2\pi kp/N} e^{-j2\pi lq/M} + \alpha_{k,-l} z_{k,-l} e^{-j2\pi kp/N} e^{j2\pi lq/M} + \alpha_{-k,-l} z_{-k,-l} e^{j2\pi kp/N} e^{j2\pi lq/M} \right\},
\]

(5.37)

where

\[
z_{\pm k, \pm l} = z_{nm, n'm'} \quad \text{for} \quad n - n' = \pm k, \quad m - m' = \pm l,
\]

(5.38)

and

\[
\alpha_{\pm k, \pm l} = \alpha_k \alpha_l,
\]

(5.39)

with

\[
\alpha_k = \begin{cases} 
\frac{N}{2} & k = 0 \\
N - |k| & \text{otherwise}
\end{cases}
\]

(5.40)

and

\[
\alpha_l = \begin{cases} 
\frac{M}{2} & l = 0 \\
M - |l| & \text{otherwise}
\end{cases}
\]

(5.41)

It is noted from (5.37) that \(\tilde{Z}_{i,i}\) can be computed by a superposition of those DFT/IDFT combinations, which can be obtained by using FFT and thus requires only \(O(NM \log NM)\) operations.

When multiple expansion modes are used to expand the current in each array element, the MoM operator matrix can be arranged such that each submatrix is a BTTB matrix as mentioned in the section 5.1. Assume that \(N_{\text{mode}}\) modes are used per element, the procedure to construct the preconditioner mentioned above can be easily extended to this case by first introducing a new Fourier operator matrix, \(F_{N_{\text{mode}}N}\), as

\[
F_{N_{\text{mode}}N} = \hat{I}_{N_{\text{mode}}} \otimes F_{NM},
\]

(5.42)

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and obtaining $\tilde{Z}$ as follows

$$
\tilde{Z} = F_{N_{\text{mode}} N M}^H N \tilde{Z} F_{N_{\text{mode}} N M}^N
\begin{bmatrix}
F_{N M}^H & F_{N M}^H & \cdots & F_{N M}^N
\end{bmatrix}
\begin{bmatrix}
\tilde{Z}^{1,1} & \tilde{Z}^{1,2} & \cdots & \tilde{Z}^{1,N_{\text{mode}}}
\tilde{Z}^{2,1} & \tilde{Z}^{2,2} & \cdots & \tilde{Z}^{2,N_{\text{mode}}}
\vdots & \vdots & \ddots & \vdots
\tilde{Z}^{N_{\text{mode}},1} & \tilde{Z}^{N_{\text{mode}},2} & \cdots & \tilde{Z}^{N_{\text{mode}},N_{\text{mode}}}
\end{bmatrix}
\begin{bmatrix}
F_{N M} & F_{N M} & \cdots & F_{N M}
\end{bmatrix}
\begin{bmatrix}
\tilde{Z}^{1,1} & \tilde{Z}^{1,2} & \cdots & \tilde{Z}^{1,N_{\text{mode}}}
\tilde{Z}^{2,1} & \tilde{Z}^{2,2} & \cdots & \tilde{Z}^{2,N_{\text{mode}}}
\vdots & \vdots & \ddots & \vdots
\tilde{Z}^{N_{\text{mode}},1} & \tilde{Z}^{N_{\text{mode}},2} & \cdots & \tilde{Z}^{N_{\text{mode}},N_{\text{mode}}}
\end{bmatrix}
= (5.43)
$$

It is worthwhile noting that the computation of each submatrix $\tilde{Z}^{i,j}$ is exactly the same as that of the one expansion mode case, and thus the diagonal elements can be obtained using (5.37). By approximating each submatrix $\tilde{Z}^{i,j}, i = 1, \ldots, N_{\text{mode}}, j = 1, \ldots, N_{\text{mode}}$ with a diagonal matrix, i.e.,

$$
\tilde{Z}^{i,j}_d = \text{diag}(\tilde{Z}^{i,j}_d),
$$

where

$$
\tilde{Z}^{i,j}_d = \text{diag}(\tilde{Z}^{i,j})
= \text{diag}(F_{N_M}^H \tilde{Z}^{i,j} F_{N_M}),
$$

$\tilde{Z}_d$ becomes a block matrix with diagonal blocks, which can be rearranged into a block diagonal matrix by introducing the appropriate permutation matrix $P$ such that

$$
\tilde{Z}_d = P\tilde{Z}_d P^T
= \begin{bmatrix}
\tilde{Z}^{1}_d & \tilde{Z}^{2}_d & \cdots & \tilde{Z}^{N_{\text{mode}}}_d
\end{bmatrix},
$$

(5.46)
is a block diagonal matrix with each diagonal block \( \hat{Z}_d^j \), \( j = 1, \ldots, \text{NM} \) being a \( N_{\text{mode}} \times N_{\text{mode}} \) matrix. It is noted that \( PP^T = P^T P = \hat{I} \) and clearly \( \hat{Z}_d = \hat{Z}_d^T \). The preconditioner in the transform domain can be defined as \( \tilde{M}^{-1} = \hat{Z}_d^{-1} \) as in the one expansion mode case, thus the preconditioner in the original domain can be given by

\[
M^{-1} = \left( P^T \hat{Z}_d^{-1} P \right)^{-1} \left( F_{N_{\text{mode}} NM}^H \right) \left( F_{N_{\text{mode}} NM} \right)
\]

where

\[
\hat{Z}_d^{-1} = \begin{bmatrix}
(\hat{Z}_d^1)^{-1} \\
(\hat{Z}_d^2)^{-1} \\
\vdots \\
(\hat{Z}_d^{\text{NM}})^{-1}
\end{bmatrix}.
\]

(5.47)

It is evident that it reduces to the one given in (5.30) if \( N_{\text{mode}} = 1 \). The preconditioned system in (5.24) can then be rewritten as

\[
\left( F_{N_{\text{mode}} NM}^H P^T \hat{Z}_d^{-1} P F_{N_{\text{mode}} NM} \right) \hat{Z}_d^i = \left( F_{N_{\text{mode}} NM}^H P^T \hat{Z}_d^{-1} P F_{N_{\text{mode}} NM} \right) \hat{v}.
\]

(5.49)

From (5.47), the matrix equation \( \mathbf{Mx} = \mathbf{y} \) can be solved by

\[
\mathbf{x} = \mathbf{M}^{-1} \mathbf{y} = \left( F_{N_{\text{mode}} NM}^H P^T \hat{Z}_d^{-1} P F_{N_{\text{mode}} NM} \right) \mathbf{y}.
\]

(5.50)

via DFT/IDFT operations and permutations without explicitly constructing matrices \( F_{N_{\text{mode}} NM} \) and \( P \).

The above development essentially constitutes the PI-MoM approach.
5.4 Numerical Results

In this section, several numerical results are presented to demonstrate the effectiveness of the PI-MoM-based full-wave solver developed in this work. As mentioned earlier, the scope of this work is to develop a solver that can predict both the radiation and scattering, respectively, from finite planar periodic arrays of four printed elements, namely, printed dipoles, microstrip-line fed patch antennas, probe-fed patch antennas, and aperture-coupled patch antennas, which are located in grounded multilayered media.

5.4.1 Arrays of Printed Dipoles

Shown in this subsection are some results pertaining to an array of printed dipoles. The dimensions of printed dipoles, the element spacings as well as the grounded multilayered medium used in the calculation are shown in figure 5.1, where the unit of length is cm. The number of expansion modes per element is 3. Figure 5.2 shows the principal plane radiation patterns of a $256 \times 256$ array of these printed dipoles, when scanned at $(\theta^s, \phi^s) = (30^\circ, 0^\circ)$ and operating at 10 GHz. The plots show patterns of both uniform and cosine excitation tapers, where the taper factors for both directions are 0.8. As can be seen, the patterns of the cosine taper exhibit lower sidelobe levels as expected. Also, figure 5.3 shows the TM bistatic RCS pattern of this printed dipole array when the incident wave comes from $(\theta^i, \phi^i) = (30^\circ, 45^\circ)$ at 10 GHz. It is worthwhile noting that the TE bistatic RCS pattern is exactly the same as the TM bistatic one, since the dipoles are thin and $\phi^i = 45^\circ$.

5.4.2 Arrays of Microstrip-line Fed Patch Antennas

This subsection presents some radiation/scattering results for some microstrip-line fed patch antenna arrays. It is noted that since a voltage current ribbon is used to model the
Figure 5.1: Geometry of the array of printed dipoles used in the calculations

feed strip line, the scattering results are obtained from arrays from patch antennas alone without feed lines. However, the open-circuit voltage of each port can be obtained from the scattering analysis, which can be used in the feed network analysis via the generalized Thevenin’s theorem. The feed network analysis will be discussed later in appendix C. The number of expansion modes per patch antenna is 3 for radiation problem and 6 for scattering problem, which is due to the fact that currents induced by the incident wave have both \( \hat{x} \) and \( \hat{y} \) components, as mentioned earlier.

Figure 5.4 shows the radiation patterns of a \( 256 \times 256 \) microstrip-line fed patch antenna array on a single-layered substrate. The dimension of the patch antenna used here is \( 4.02 \times 4.02 \text{ cm} \) and the width of the feed line is \( 0.445 \text{ cm} \). The spacings in both directions are \( 5 \text{ cm} \) and the operating frequency is \( 2.28 \text{ GHz} \). The substrate is \( 0.159 \text{ cm} \) thick and its dielectric constant is \( 2.55 - j0.0051 \). The plots show patterns of both uniform and exponential excitation tapers, where the taper factors for both directions are 1.0. It is evident from the figure that tapering the excitation reduces the sidelobe levels as is well known.
Likewise, both TE and TM bistatic RCS patterns of a $256 \times 256$ patch antenna array on a single-layered substrate are plotted in figure 5.5. The dimension of the patch antenna used here is $(L, W) = (3.66, 2.6)$ cm. and the spacings in both directions are 5.5517 cm. The substrate is 0.158 cm. thick and its dielectric constant is 2.17. The incident wave comes from $(\theta^i, \phi^i) = (30^\circ, 0^\circ)$ at 3.7 GHz. It can be seen from the figure that this array exhibits a grating lobe at around $76^\circ$ in the E-plane. Also, figure 5.6 shows both TM and TE monostatic RCS patterns of this patch antenna array, which exhibit a peak at around $47^\circ$ due to the grating lobe.
Figure 5.3: TM bistatic RCS patterns of a $256 \times 256$ printed dipole array for $(\theta^i, \phi^i) = (30^\circ, 45^\circ)$.

Figure 5.4: Radiation patterns of a $256 \times 256$ microstrip-line fed patch antenna array in principle planes. $\theta^s = 30^\circ$, $\phi^s = 0^\circ$. 
Figure 5.5: TE and TM bistatic RCS patterns of a $256 \times 256$ patch antenna array for $(\theta^i, \phi^i) = (30^\circ, 0^\circ)$.

Figure 5.6: TM and TE monostatic RCS patterns of a $256 \times 256$ patch antenna array in the $\phi^i = 0^\circ$ plane.
5.4.3 Arrays of Probe-fed Patch Antennas

This subsection presents some radiation/scattering results from some probe-fed patch antenna arrays based on the PI-MoM approach. It is noted that the probe is modelled using the wire modes and the total number of wire modes depends on the number of layers in which the probe is located. Also, the number of wire modes is determined based on the thickness of each layer, hence the number of wire modes in each layer segment may be different. The patch currents are expanded using both $\hat{x}$-directed and $\hat{y}$-directed patch modes, which are subdivided more finely that those for the microstrip-line fed patch antenna, and the attachment mode is used to enforce the continuity of the current at the probe-patch junction. Therefore, the number of expansion modes for each probe-fed patch antenna is quite large as compared to those for printed dipole and microstrip-line fed patch antenna. Typically, the minimum number of expansion modes per element here is $12 + 12 + 2 + 1 = 27$. Note that for probe-fed patch antenna arrays, both radiation and scattering computations use the same set of expansion modes.

The first example is a $128 \times 128$ probe-fed patch antenna array on a single-layered substrate. The dimension of the patch antenna is $(L,W) = (2.0, 3.0)$ cm. and the spacings in both directions are 4.0 cm. The probes are located 0.65 cm. from the edge of each patch and its inner, outer radii are 0.043, 0.14, cm. respectively. The substrate is 0.127 cm. thick and its dielectric constant is $10.2 - j0.051$. Figure 5.7 shows the radiation patterns when scanned at $(\theta^e, \phi^e) = (30^\circ, 0^\circ)$ and operating at 2.3025 GHz. The plots show patterns of both uniform and exponential excitation tapers, where the taper factors for both directions are 1.0, and it can be seen that the sidelobe levels can be reduced by tapering the excitation. Likewise, both TE and TM bistatic RCS patterns of this probe-fed patch antenna array at the same frequency are shown in figure 5.8. Similar to the case
of microstrip-line fed patch antenna arrays, the short-circuit current of each port can be obtained from the scattering analysis, which can be used in the feed network analysis via the generalized Norton’s theorem.

![E-plane pattern of 128×128 probe-fed patch antenna array, \((\theta^s, \phi^s) = (30^\circ, 0^\circ)\)](image)

![H-plane pattern of 128×128 probe-fed patch antenna array, \((\theta^s, \phi^s) = (30^\circ, 0^\circ)\)](image)

Figure 5.7: Radiation patterns of a 128 × 128 probe-fed patch antenna array in principle planes. \(\theta^s = 30^\circ, \phi^s = 0^\circ\).

The second example is a 128 × 128 probe-fed patch antenna array on a single-layered substrate and covered by a single-layered superstrate. The patch antenna geometry as well as the multilayered medium are shown in figure 5.9 with the unit of length being cm. The spacings \((d_x, d_y) = (10.0, 12.0)\) cm. and the inner, outer radii of the probe are 0.0635, 0.2, cm. respectively. Figure 5.10 shows the radiation patterns when scanned at \((\theta^s, \phi^s) = (30^\circ, 0^\circ)\) and operating at 1.14 GHz. Likewise, both TE and TM bistatic RCS patterns of this probe-fed patch antenna array at the same frequency are shown in figure 5.11. Note that the number of expansion modes per element for this array is also 27.
The third example is a $64 \times 64$ probe-fed patch antenna array on a double-layered substrate and covered by a single-layered superstrate. The patch antenna geometry as well as the multilayered medium are shown in figure 5.12 with the unit of length being cm. The spacings in both directions are 6.0 cm. and the inner, outer radii of the probe are 0.043, 0.14, cm. respectively. Figure 5.13 shows the radiation patterns when scanned at $(\theta^i, \phi^i) = (30^\circ, 0^\circ)$ and operating at 2.33 GHz. Likewise, both TE and TM bistatic RCS patterns of this probe-fed patch antenna array at the same frequency are shown in figure 5.14. Note that the number of expansion modes per element for this array is 29 since the probes extend to two layers and the total number of wire modes per element becomes 4.
Figure 5.9: Geometry of patch antenna and multilayered medium used in the second example.

\[
\begin{align*}
\varepsilon_{r1} &= 2.64 - j0.00792 \\
\varepsilon_{r2} &= 13.2 - j0.0396 \\
\varepsilon_{r3} &= 1.52 \\
\varepsilon_{r3} &= 1
\end{align*}
\]

Figure 5.10: Radiation patterns of a 128 × 128 probe-fed patch antenna array in principle planes. \( \theta^s = 30^\circ, \phi^s = 0^\circ \).
Figure 5.11: TE and TM bistatic RCS patterns of a 128 \times 128 probe-fed patch antenna array for \((\theta^i, \phi^i) = (30^\circ, 0^\circ)\).

Figure 5.12: Geometry of patch antenna and multilayered medium used in the third example

\[
\begin{align*}
  \varepsilon_{r4} &= 1 \\
  \varepsilon_{r3} &= 4.25 - j0.004 \\
  \varepsilon_{r2} &= 8.5 - j0.0085 \\
  \varepsilon_{r1} &= 10.2 - j0.051
\end{align*}
\]
Figure 5.13: Radiation patterns of a 64 × 64 probe-fed patch antenna array in principle planes. \( \theta^a = 30^\circ, \phi^a = 0^\circ \).

Figure 5.14: TE and TM bistatic RCS patterns of a 64 × 64 probe-fed patch antenna array for \((\theta^i, \phi^i) = (30^\circ, 0^\circ)\).
5.4.4 Arrays of Aperture-coupled Patch Antennas

Shown in this subsection are some results regarding radiations from some aperture-coupled patch antenna arrays. It is noted that the scattering of these arrays can be obtained with aperture closed resulting in the exactly same calculation of the scattering from microstrip-line fed patch antenna arrays. The short-circuit current on each aperture can be found from the scattering analysis, which can be used in the feed network analysis via the generalized Norton’s theorem. The patch currents for these array elements are expanded using both \( \hat{x} \)-directed and \( \hat{y} \)-directed patch modes in the same way as done for the probe-fed patch antenna arrays. Typically, the number of expansion modes per element here is \( 12 + 12 = 24 \).

The first example is a \( 128 \times 128 \) aperture-coupled patch antenna array on a single-layered substrate. The dimension of the patch antenna is \( (L, W) = (2.0, 3.0) \) cm. and the spacings in both directions are 4.0 cm. The apertures are located just below the center of each patch and their dimensions are \( 1.12 \times .11 \) cm. The substrate is 0.127 cm. thick and its dielectric constant is \( 10.2 - j0.0102 \). Figure 5.15 shows the radiation patterns when this array is scanned at broad side and operates at 2.3025 GHz.

The second example is a \( 128 \times 128 \) aperture-coupled patch antenna array on a single-layered substrate and covered by a single-layered superstrate as the one used as the second probe-fed patch antenna array example. The patch antenna geometry as well as the multi-layered medium are shown in figure 5.9 with the unit of length being cm., except that the probe is replaced by the slot with dimension \( 5.0 \times .5 \) cm. and located just below the center of the patch. The spacings \( (d_x, d_y) = (10.0, 12.0) \) cm. Figure 5.16 shows the radiation patterns when scanned at \( (\theta^s, \phi^s) = (0^\circ, 0^\circ) \) and operating at 1.14 GHz.
E-plane pattern of 128×128 aperture-coupled patch antenna array, $(\theta^s, \phi^s) = (0^\circ, 0^\circ)$

(a) E-plane

H-plane pattern of 128×128 aperture-coupled patch antenna array, $(\theta^s, \phi^s) = (0^\circ, 0^\circ)$

(b) H-plane

Figure 5.15: Radiation patterns of a 128 × 128 aperture-coupled patch antenna array in principle planes. $\theta^s = 0^\circ$, $\phi^s = 0^\circ$.

E-plane pattern of 128×128 aperture-coupled patch antenna array, $(\theta^s, \phi^s) = (0^\circ, 0^\circ)$

(a) E-plane

H-plane pattern of 128×128 aperture-coupled patch antenna array, $(\theta^s, \phi^s) = (0^\circ, 0^\circ)$

(b) H-plane

Figure 5.16: Radiation patterns of a 128 × 128 aperture-coupled patch antenna array in principle planes. $\theta^s = 0^\circ$, $\phi^s = 0^\circ$. 
CHAPTER 6

PI-MOM DEVELOPMENT FOR ARRAYS WITH NON-RECTANGULAR ELEMENT TRUNCATION BOUNDARIES

The finite periodic arrays discussed so far are assumed to have rectangular element truncation boundaries, i.e., an $M \times N$ array consists of $N$ rows with $M$ elements in every row. As discussed earlier, the MoM operator matrices for arrays with rectangular boundaries are BTTB due to the periodicity of the array elements. This property is highly preferable because only elements in first few rows are required to be evaluated, while other elements can be constructed from these few rows. Thus, both computational cost and memory storage can be significantly reduced. Moreover, a fast matrix-vector multiplication can be carried out in terms of a fast Fourier transform (FFT), which accelerates the iteration process. However, arrays in practical applications may also have non-rectangular element truncation boundaries, such as circular or hexagonal arrays; thus it is desirable to extend the preconditioned iterative solvers developed in this work to analyze these kinds of arrays as well.

This chapter will discuss the extension of the PI-MoM method developed in this work to handle finite periodic arrays with non-rectangular element truncation boundaries, which will later be referred to as non-rectangular arrays, as depicted in figure 6.1. Section 6.1 will discuss the array shape matrix and the shaping operator which will be used to obtain
the MoM operator matrix for non-rectangular arrays from the MoM operator matrix for the enclosing rectangular arrays. The application of the array shape matrix in the PI-MoM based preconditioned iterative solvers will be discussed in section 6.2 followed by some numerical results given in section 6.3.

![Figure 6.1: A finite antenna array with non-rectangular element truncation boundary in a multilayered medium](image)

### 6.1 Array Shape Matrix

Consider first the case when only one expansion mode is used to represent the equivalent current distribution on each array element. It is well-known that the MoM operator matrix for an \( M \times N \) element finite periodic array, denoted by \( \mathbf{Z} \), becomes a \( NM \times NM \) block Toeplitz with Toeplitz blocks (BTTB) matrix. Here, \( M \) and \( N \) are integers. However,
if the number of elements in each row are not the same, the Toeplitz blocks will not be uniform, and thus the BTTB structure will be destroyed. This is highly undesirable since the MoM operator matrix cannot be represented by only one row as typically be the case for finite planar periodic arrays, and furthermore the fast matrix-vector multiplication method using FFT also cannot be applied.

Now, let $Z_A$ be the MoM operator matrix for the original finite array with a non-rectangular element truncation boundary, which is circumscribed by a somewhat larger $M \times N$ rectangular boundary. Assume that the total number of elements in the original non-rectangular array is $K$, then obviously $K < NM$. It is evident that $Z_A$ is a $K \times K$ non-BTTB matrix and it must contain information regarding the element truncation boundary. However, one can observe that all elements in $Z_A$ must be included in $Z$, which is the MoM operator matrix for the circumscribing $M \times N$ rectangular array. In other words, $Z_A$ can be constructed from $Z$.

The observation made above leads to the introduction of an array shape matrix $A$ which is an $NM \times K$ matrix which contains the element truncation boundary information. Its elements are defined as

$$a_{nm,k} = \begin{cases} 1 & \text{for } \text{index}(k) = nm \\ 0 & \text{otherwise} \end{cases},$$

(6.1)

where the function $\text{index}(k)$ maps the $k^{th}$ index in the non-rectangular array into the corresponding double index $(n,m)$ for the equivalent circumscribing rectangular array or in other words, it relates the $k^{th}$ element in the non-rectangular array to the $(n,m)^{th}$ element in the rectangular array. Figure 6.2 shows an example of an octagonal array and its element indices. Also shown are the double indices used for the conventional $M \times N$ rectangular array which encloses (or circumscribes) the octagonal array. For the example shown in that figure, $K = 36, M = 8, N = 6$, and index(1) = 13, index(15) = 35, index(36) = 66, etc.
Using the array shape matrix, one can obtain \( Z_A \) from \( Z \) via the relation

\[
Z_A = A^T Z A_{NM \times NM} (\cdot) A_{K \times K}. \tag{6.2}
\]

The operation \( A^T Z A \) simply represents the selection of appropriate rows and columns in \( Z \) to construct a new matrix according to the information specified in the array shape matrix, and thus \( A^T (\cdot) A \) can be considered a ‘shaping’ operator. It is noted that \( A^T A = \hat{I}_K \), where \( \hat{I}_K \) is the \( K \times K \) identity matrix.

![Example of an array with non-rectangular element truncation boundary](image)

Figure 6.2: Example of an array with non-rectangular element truncation boundary; numbers indicate the element indices for both arrays.

The approach mentioned above can be easily extended to the case of multiple expansion modes by first noticing that the MoM matrix operator matrix in this case becomes a block matrix with each block being a BTTB matrix. Thus, by applying the shaping operator \( A^T (\cdot) A \) given in (6.2) to each BTTB submatrix, the MoM operator matrix for the non-rectangular array can be obtained. Now, introducing \( A_{N_{\text{mode}}} \) such that

\[
A_{N_{\text{mode}}} = \hat{I}_{N_{\text{mode}}} \otimes A, \tag{6.3}
\]
where $N_{\text{mode}}$ denotes the number of expansion modes per array element, and the operator $\otimes$ denotes the Kronecker product, then the MoM operator matrix for the non-rectangular array using multiple expansion modes can be given by

$$
Z_A = A^T N_{\text{mode}} A N_{\text{mode}} = \begin{bmatrix}
Z_{1,1} & Z_{1,2} & \cdots & Z_{1,N_{\text{mode}}} \\
Z_{2,1} & Z_{2,2} & \cdots & Z_{2,N_{\text{mode}}} \\
\vdots & \vdots & \ddots & \vdots \\
Z_{N_{\text{mode}},1} & Z_{N_{\text{mode}},2} & \cdots & Z_{N_{\text{mode}},N_{\text{mode}}} 
\end{bmatrix},
$$

(6.4)

where

$$
Z_{i,j}^{A} = A^T Z_{i,j}^{A} A.
$$

(6.5)

Clearly, the $K \times K$ submatrix $Z_{i,j}^{A}$ represents the coupling between the $i^{th}$ and the $j^{th}$ expansion modes.

It is evident from (6.2) and (6.4) that by introducing the array shape matrix, the block-Toeplitz property of the MoM operator matrix can still be exploited and therefore the storage requirement can be kept minimal. Also, it is worthwhile noting that the shaping operator can be applied without explicitly constructing the matrix $A$. It only needs to store the information regarding element truncation boundary, which does not require large amount of memory.

### 6.2 Application of the Array Shape Matrix in Iterative Solvers

It is well-known that the computational cost of iterative solvers is proportional to the cost of matrix-vector multiplication. For an $NM \times NM$ BTTB matrix, the fast matrix-vector multiplication method using FFT can be applied to compute a matrix-vector product, which requires only $O(NM \log NM)$ operations instead of $O((NM)^2)$ operations. Since the MoM operator matrix for a $M \times N$ finite planar periodic array becomes a BTTB matrix.
for one expansion mode case and a block matrix with BTTB blocks for the case of multiple
expansion modes, this method can be applied to improve the efficiency of iterative solvers.
By using the array shape matrix defined in the previous subsection, it is evident that this
fast multiplication method can be applied to non-rectangular array problems as follows.
For the one expansion mode case, using (6.2) yields

\[ Z_A i = A^T Z A i \]
\[ = A^T (Z y), \]

(6.6)

where \( y = A i \). Since \( Z \) is a BTTB matrix, the matrix-vector product in the bracket can
be computed using FFT. It is noted that the matrix \( A \) has only small number of non-zero
elements, thus the cost of its multiplication will be extremely minimal. This approach can
be easily extended to the case of multiple expansion modes by noticing that the matrix-
vector of each submatrix \( Z_A^{i,j} \) can be computed in the same way as

\[ Z_A^{i,j} i_j = A^T Z^{i,j} A i_j \]
\[ = A^T (Z^{i,j} y_j), \]

(6.7)

where \( i_j \) is the coefficient vector for the \( j^{th} \) mode and \( y_j = A i_j \). Therefore, the fast matrix-
vector multiplication using FFT is also applicable for the non-rectangular array problems
with additional shaping operations that are not expensive.

In general, the efficiency of iterative solvers depends greatly on the convergence rate
of their solutions, which in turn is a function of the condition number of the pertaining
matrix. In this work, a DFT-based preconditioner developed in [49] is implemented to
accelerate the convergence of iterative solvers. This preconditioner is considered effective
since it is a good approximation to a block matrix with BTTB blocks and can be computed
efficiently by FFT without explicitly constructing large matrices. As discussed earlier, the
MoM operator matrix for the non-rectangular arrays can be obtained by simply applying the appropriate shaping operator to the matrix for the enclosing rectangular array. This observation leads to the choice of a preconditioner matrix, $M_A^{-1}$, for the non-rectangular arrays, where

$$
M_A^{-1} = A^TM^{-1}A,
$$

and $M^{-1}$ is the preconditioner for the rectangular array. It can be observed that if the number of elements in the non-rectangular arrays is comparable to the number of elements in the enclosing rectangular array, i.e., $K \approx NM$, the array shape matrix will approach the identity matrix, thus $M_A^{-1} \approx M^{-1}$. The preconditioned system for the one expansion mode case can be given as follows:

$$
\left(\begin{array}{c}
A^TF_{NM}^H\tilde{Z}_d^{-1}F_{NM}A \\
M_A^{-1}
\end{array}\right)\tilde{i} = \left(\begin{array}{c}
A^TF_{NM}^H\tilde{Z}_d^{-1}F_{NM}A \\
Z_A
\end{array}\right)v,
$$

(6.9)

where $F_{NM}$ denotes the two-dimensional FFT matrix. It is noted that $AA^TZ$ on the left hand side of (6.9) will only ‘remove’ certain rows and columns which correspond to elements not included in the non-rectangular arrays from $Z$, and thus $M^{-1}AA^TZ$ should have the spectral properties comparable to those of $M^{-1}Z$, which in turn makes the spectral properties of $M_A^{-1}Z_A$ comparable to those of $M^{-1}Z$. Therefore, this modified preconditioned system should work reasonably well compared to the original one. Finally, the modified preconditioned system for the multiple expansion modes case can be simply obtained by using $A_{N_{mode}}$ instead of $A$ in (6.9). From (5.31), it can be given by

$$
M_A^{-1}Z_A\tilde{i} = M_A^{-1}v,
$$

(6.10)
where
\[
M_A^{-1} = A_{N_{mode}}^{T} F_{N_{mode},NM}^H Z_d^{-1} F_{N_{mode},NM} \tilde{A}_{N_{mode}}
\]
\[
= A_{N_{mode}}^{T} F_{N_{mode},NM}^H P^T Z_d^{-1} P F_{N_{mode},NM} \tilde{A}_{N_{mode}}
\]
\[
= A_{N_{mode}}^{T} M^{-1} A_{N_{mode}}
\]
(6.11)
\[
Z_A = A_{N_{mode}}^{T} Z A_{N_{mode}}
\]
(6.12)
and \(M^{-1}\) is the preconditioner for the rectangular array as before. This completes the PI-MoM development for arrays with non-rectangular element truncation boundaries.

### 6.3 Numerical Results

In this section, several numerical results are presented, based on the extension of PI-MoM to treat non-rectangular arrays, to illustrate the effectiveness of the approach described above. The basis functions used in this work are based on the overlapping piecewise sinusoidal basis function.

The first example deals with the radiation from printed dipole arrays in a grounded multilayered medium with hexagonal and rectangular array boundaries. The elements and substrate as well as the array element spacings used in this calculations are shown in figure 6.3. Since the printed dipole is assumed to be thin, only the \(\hat{x}\)-directed current is excited; thus, only \(\hat{x}\)-directed expansion modes are required. Three expansion modes per one element are used in this calculation. The hexagonal truncation boundary is shown in figure 6.4, which is circumscribed or enclosed by an \(100 \times 87\) element rectangular truncation boundary, and the total number of elements in the hexagonal array is 6,500, while that of the circumscribing array is 8,700. Figure 6.5 shows the calculation of radiation from the array of printed dipoles with the hexagonal truncation boundary and also for the reference
purpose, the radiation from the rectangular array which encloses the hexagonal array. As can be seen from the figure, the radiation pattern of the hexagonal array has lower sidelobes in the principal plane ($\phi = 0^\circ$ plane) while it has higher sidelobes in the $\phi = 45^\circ$ plane. It is noticed that in the $\phi = 0^\circ$ plane, the hexagonal array has no edge while the rectangular array has two, which leads to more edge diffraction effects which in turn results in higher sidelobes. It is also noted that the opposite occurs in the $\phi = 45^\circ$ plane.

![Printed dipole array](image)

Figure 6.3: Geometry of the array of printed dipoles used in the calculations

The second example deals with the plane wave scattering from patch antenna arrays on a grounded single-layered medium with elliptic and rectangular boundaries. The dimension of the patch antenna used here is $(L, W) = (3.66, 2.6)$ cm. and the spacings in both directions are 5.5517 cm. The substrate is 0.158 cm. thick and its dielectric constant is 2.17. The incident plane wave comes from $(\theta^i, \phi^i) = (30^\circ, 0^\circ)$ at 3.7 GHz. Since both $\hat{x}$-directed and $\hat{y}$-directed currents are excited by the incident field, both $\hat{x}$-directed and $\hat{y}$-directed expansion modes are needed and thus the number of expansion modes per element used here is 6. The elliptic truncation boundary is shown in figure 6.6, which is enclosed by a $200 \times 100$ boundary, and the total number of elements in the elliptic array is 15,708, while that in the enclosing array is 20,000. Figure 6.7 shows TM bistatic RCS pattern in

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the E-plane of these patch antenna arrays with elliptic and rectangular element truncation boundaries. As can be seen from the figure, the RCS patterns of both elliptic and rectangular arrays have almost the same level of peaks, but the elliptic array has significantly lower sidelobes, which is the same trend seen in the radiations from printed dipole arrays with hexagonal and rectangular boundaries.

The last example is the radiation from probe-fed patch antenna arrays on a single-layered substrate with circular and square boundaries. The dimension of the patch antenna is \((L, W) = (2.0, 3.0)\) cm. and the spacings in both directions are 4.0 cm. The probe feeds are located 0.65 cm. from the edge of each patch and its inner, outer radii are 0.043, 0.14, cm. respectively. The substrate is 0.127 cm. thick and its dielectric constant is \(10.2 - j0.051\). The circular boundary is shown in figure 6.8, which is circumscribed or enclosed by a \(128 \times 128\) square boundary, and the total number of elements in the circular array is 12,892, while that in the circumscribing array is 16,384. The unknown current for each

Figure 6.4: Hexagonal array element truncation boundary used in the calculation
Figure 6.5: Radiation from an array of printed dipoles with hexagonal array element truncation boundary shown in figure 6.4 compared with that from the enclosing rectangular array

Element is expanded by \( \hat{x} \)-directed patch modes, \( \hat{y} \)-directed patch modes, wire modes and an attachment mode to enforce the continuity of the current at the probe-patch junction. The total number of expansion modes per element is 27. The square array is excited by an exponential excitation taper, while the circular array is excited by the uniform taper. Figure 6.9 shows the E-plane radiation patterns of both arrays when scanned at \((\theta^s, \phi^s) = (30^\circ, 0^\circ)\) and operating at 2.3025 GHz. As can be seen, the circular array has lower sidelobes and thus in certain cases shaping of array element truncation boundaries can lower the sidelobe levels in the principal plane.
Figure 6.6: Elliptic array element truncation boundary used in the calculation

Figure 6.7: TM bistatic RCS pattern of patch antenna arrays with elliptic and rectangular boundaries. The incident plane wave comes from $(\theta^i, \phi^i) = (30^\circ, 0^\circ)$. 
Figure 6.8: Circular array element truncation boundary used in the calculation

Figure 6.9: E-plane radiation patterns of probe-fed patch antenna arrays with circular and square boundaries. The scan angle is \((\theta^s, \phi^s) = (30^\circ, 0^\circ)\).
Although the PI-MoM approach utilizing a preconditioned iterative solver which has been developed in this work can solve most MoM matrix equations quite efficiently, it may still encounter a convergence difficulty due to a large condition number of the MoM operator matrix. This problem typically occurs when either the total thickness of the multilayered medium becomes electrically large or the dielectric constant of the medium is large. An alternative, hybrid UTD-MoM approach for solving the MoM matrix equation is developed here to overcome this PI-MoM convergence problem which may occur for some cases as indicated above.

The hybrid UTD-MoM method developed here can be considered an extension to the approach developed in [51] for analyzing large finite planar arrays of printed dipoles in free space. The basic idea leading to this approach comes from the ray analysis of the fields radiated by finite planar arrays on grounded media. It is evident that the ray fields from infinite arrays consist of only Floquet-type wave and the finiteness of arrays induces the edge-diffracted and corner-diffracted fields as well as the edge-excited and corner-excited surface and leaky waves. By using the results from the ray analysis, one can relate one
unknown MoM current coefficient to another, but sufficiently far from array element truncation boundary edges and corners, in terms of newly-introduced UTD-based basis functions; this relation leads to a drastic reduction in the number of unknowns needed to be solved. Since the number of unknowns can be limited to quite a small number in this hybrid approach, a direct matrix solver can be used to solve the new matrix equation without significantly increasing the computational cost; thus, it can help eliminate the convergence problem due to a large condition number. This is highly advantageous because it can be implemented without changing MoM basis functions and it is expected to work well with any subsectional basis function for the array element currents.

This chapter consists of four major parts. First, the results from a DFT-based UTD ray analysis of large finite phased arrays on grounded media will be presented in section 7.1. Section 7.2 will describe the hybrid UTD-MoM approach which is based on the results discussed in section 7.1 for arrays with rectangular element truncation boundaries. The method can be extended to arrays with non-rectangular element truncation boundaries as well, as discussed in section 7.3. Finally, some numerical results based on the hybrid UTD-MoM will be presented in section 7.4.

7.1 A DFT-based UTD Ray Analysis

By employing the DFT representation for the array current distribution, the composite array fields radiated by each DFT component, which is a simple uniform distribution with a linear phase, can be efficiently expressed in terms of asymptotic closed form expressions that provide the UTD ray fields as follows [55, 70].
7.1.1 Floquet-type modal wave

The Floquet-type wave is a modal sum of plane waves representing the fields produced by an infinite periodic array. For a given frequency, only a finite number of lower order modes propagate away from the array face while the remaining infinite number of modes are non-propagating or evanescent above the array. For a finite array, the Floquet modal waves are shadowed by the array edges, thus they exist only within a limited region of space [51, 54].

7.1.2 Floquet edge-diffracted field

Since the Floquet waves (FWs) are abruptly truncated at the array edges, they undergo diffraction at the edges yielding the Floquet edge-diffracted fields. The diffracted field for each FW mode lies on a Keller cone of diffraction, whose half cone angle is defined as usual by the phase matching condition at the edge. The Floquet edge-diffracted field is in the form of a conical wave emanating from the edge of the array. It has a propagation constant projected along the edge which is matched to the FW propagation constant of the mode that is undergoing the edge diffraction. The Floquet edge-diffracted field smoothly compensates the discontinuity of the associated Floquet mode which suddenly vanishes at the shadow boundary plane arising from that edge. The Floquet edge-diffracted field also consists of both propagating and evanescent modes. Note that the Floquet edge diffracted fields also exist in a restricted region of space since the edges are finite. The shadow boundaries for Floquet edge diffracted fields are cones that arise at the corners into which edges terminate.
7.1.3 Floquet corner-diffracted field

The Floquet corner-diffracted field is in the form of a spherical wave emanating from the corner of the array. It arises from the diffraction of the FW modes due to the element truncation of the array at the corners. The Floquet corner-diffracted field smoothly compensates the discontinuity of the Floquet edge-diffracted fields at the corresponding shadow boundary cones arising at the corner, and it also compensates the discontinuity of FW as well as Floquet edge diffracted fields simultaneously in situations where there is an overlap of four shadow boundaries. The latter can occur when the two shadow boundary planes of the Floquet modal wave and the two shadow boundary cones of the edge-diffracted fields coincide.

7.1.4 Edge-excited surface and leaky waves

Surface and leaky waves are launched from the array edges due to the Floquet edge diffraction discussed earlier. These complex waves are the same as those that can exist on the grounded substrate in the absence of the array. There is no spreading of these edge-excited complex waves and their propagation constant is matched to the Floquet wave number in the direction along the edge. The edge-excited complex waves exist in a certain region as do the Floquet edge-diffracted fields and suddenly disappear at the shadow boundary plane due to the truncation of the edge. In general, more than one surface wave mode can be excited depending on the configuration of the slab. In some configurations, the leaky waves can also be excited in addition to the surface waves.
7.1.5 Corner-excited surface and leaky waves

Corner-excited surface and leaky waves are launched from the array corners due to the Floquet corner diffraction discussed previously. Similarly, these complex waves are the same as those that can exist on the grounded substrate in the absence of the array. The corner-excited complex waves have a cylindrical spreading factor along the surface. The corner-excited complex waves smoothly compensate the discontinuities of the corresponding edge-excited complex waves at the shadow boundary planes arising at the array edges as discussed earlier.

The collective ray field representation discussed above is relatively simple, efficient, and physically appealing, and thus it can be used to relate one unknown current coefficient to another current coefficient, which will be discussed in the next section.

7.2 A Hybrid UTD-MoM

Based on the results of the UTD ray analysis of finite planar phased arrays on grounded media given in the previous section, the array field can be given in terms of Floquet-type (FW) modal wave and its edge and corner diffractions as well as the edge-excited and corner-excited surface and leaky waves. Therefore, by introducing new set of basis functions which represent these field components, one can represent the unknown MoM current coefficients in terms of these functions. Since it generally requires only small number of these functions and the number does not depend on the number of array elements, the number of new unknowns can be drastically reduced and does not change regardless of the number of elements. Hence, it is highly efficient for large arrays.
Assuming that the total of propagating Floquet modal waves for a particular finite-sized array is \( J \), the total electric field at an observation point can be expressed as

\[
E \sim \sum_{j=1}^{J_1} E_{jW}^{FW} + \sum_{k=1}^{J_2} \sum_{j=1}^{J_2} E_{jW}^{ed,k} + \sum_{k=1}^{J_3} \sum_{j=1}^{J_3} E_{jW}^{cd,k} + \sum_{k=1}^{J_4} \sum_{j=1}^{J_4} E_{jW}^{esw,k} + \sum_{k=1}^{J_5} \sum_{j=1}^{J_5} E_{jW}^{csw,k},
\]

(7.1)

where \( E_{jW}^{FW} \) is the \( j^{th} \) Floquet modal wave, \( E_{jW}^{ed,k} \) and \( E_{jW}^{cd,k} \) denote the \( k^{th} \) edge and the \( k^{th} \) corner diffracted rays, respectively, and \( E_{jW}^{esw,k} \) and \( E_{jW}^{csw,k} \) denote the edge and corner excited surface waves, respectively, all produced by \( E_{jW}^{FW} \). \( J_i \) denotes the number of Floquet modal waves for each type of ray. For instance, \( J_1 \) is the number of Floquet modal wave, while \( J_3 \) is the number of Floquet corner diffracted rays. \( J_{\{1,2,3,4,5\}} \leq J \) since all the Floquet modal waves as well as edge/corner diffracted and/or edge/corner excited surface waves of all Floquet modal waves may not reach the observation point.

Figure 7.1 shows a typical UTD ray picture of finite planar arrays when the number of propagating Floquet modal wave is one. As mentioned previously, the edge and corner diffractions as well as the edge-excited and corner-excited surface and leaky waves result from the array truncation. Thus, with this ray picture in mind, the behavior of unknown current coefficients, denoted here by \( i_{nm} \), in the array interior can be approximated by FW modes pertaining to the infinite array solution, together with the modifications resulting from the effects of the FW diffraction at edges and corners arising from the finiteness of the arrays. The edge and corner diffraction effects as well as the surface wave effects are assumed to have the same functional forms as the fields. Hence, it is convenient to divide the array into a large inner part, and small remaining outer parts as depicted in figure 7.2. As can be seen from the figure 7.2, the outer part consist of four edge parts and four corner parts. First, the whole array is broken up into a collection of individual periodic (unit) cells in which each \((nm)^{th}\) cell contains the \((nm)^{th}\) array element. Each edge part is confined
to a distance of three cells from the actual edge of the array, whereas each corner part is
confined to a region filled by fifteen cells. The region within the physical array which
excludes the edge parts and corner parts defines the remaining, much larger, inner part.
It is typically found that the observation point needs to be one or two wavelengths away
from the array edges and corners for the asymptotic high frequency based UTD functional
forms of the fields to be reasonably accurate; this information indicates the size of the outer
dge and corner regions for the hybrid solution and is independent of the electrical size of
the array. Thus, one can now specifically describe the functional variation of $i_{nm}$ using a
global, UTD-based approximation, as follows:

$$i_{nm} = \begin{cases} 
  i_{inner} & \text{for the inner part} \\
  i_{edge} & \text{for the edge parts} \\
  i_{corner} & \text{for the corner parts}
\end{cases}$$

(7.2)

$i_{corner}$ in the above equation is exactly the same as the original MoM current coefficient,
whereas $i_{inner}$ and $i_{edge}$ are given in terms of new UTD-based unknown coefficients. To
reduce the complexity of solution, it is assumed in this work that the effects from corner
diffracted rays are small compared to those from edge diffracted rays and edge-excited sur-
face waves, which is reasonable for any $(nm)$th cell farther than 2 wavelengths away from
an array corner. Since antenna arrays are typically designed to have only one propagating
Floquet modal wave, the number of Floquet modal wave used here is restricted to 1. Taking
into account all considerations mentioned above, $i_{inner}^{nm}$ can be expressed as

$$i_{inner}^{nm} \approx j^{FW} f^{FW}(n, m) + \sum_{k=1}^{4} \left[ \frac{j_{k,1}^{ed}}{s_{e,k}} + \frac{j_{k,2}^{ed}}{s_{e,k}^3} + \frac{j_{k,3}^{ed}}{s_{e,k}^5} \right] f^{ed}(n, m, k)$$

$$+ \sum_{k=1}^{4} j_{k,1}^{esw} f^{esw}(n, m, k)$$

$$+ \sum_{k=1}^{4} \left[ j_{k,x}^{csw,x} f^{csw,x}(n, m, k) + j_{k,y}^{csw,y} f^{csw,y}(n, m, k) \right],$$

(7.3)
where $j^{iFW}$, $j^{ed}_{k,l}$, $k \in \{1, 2, 3, 4\}$, $l \in \{1, 2\}$, and $j^{esw}_{k}$, $k \in \{1, 2, 3, 4\}$ are the new unknown coefficients. Since the terms including $f^{csu,x}$ and $f^{csu,y}$ are introduced here to compensate the discontinuities of the corresponding edge-excited surface wave in the shadow regions, the unknowns associated with $f^{csu,x}$ and $f^{csu,y}$ are related to $j^{esw}$ as

$$j^{csu,x}_{k} = \begin{cases} j^{esw}_{2} & \text{for corners 1, 2} \\ j^{esw}_{4} & \text{for corners 3, 4} \end{cases} \tag{7.4}$$

and

$$j^{csu,y}_{k} = \begin{cases} j^{esw}_{1} & \text{for corners 1, 4} \\ j^{esw}_{3} & \text{for corners 2, 3} \end{cases} \tag{7.5}$$

The function $f^{FW}$ in (7.3) is a functional form for the Floquet modal wave, which can be written as

$$f^{FW}(n, m) = V_{nm} e^{-j\beta_{x}x_{m}} e^{-j\beta_{y}y_{n}}, \tag{7.6}$$

where $(x_{m}, y_{n})$ and $V_{nm}$ denote the center and the feed voltage amplitude, respectively, of the $(nm)^{th}$ cell. Also, $\beta_{x}$ and $\beta_{y}$ are the impressed phase defined as

$$\beta_{x} = k_{0} \sin \theta \cos \phi, \tag{7.7}$$

$$\beta_{y} = k_{0} \sin \theta \sin \phi, \tag{7.8}$$

where $(\theta, \phi)$ specifies the scan direction of the main beam and $k_{0}$ is the free-space propagation constant. Likewise, the function $f^{ed}$ is a functional form for the edge diffracted ray, which can be expressed as

$$f^{ed}(n, m, k) = V(x_{e,k}, y_{e,k}) e^{-jk_{0}s_{e,k}} e^{-j\beta_{x}x_{e,k}} e^{-j\beta_{y}y_{e,k}} U_{e}(x_{e,k}, y_{e,k}), \tag{7.9}$$

where $V(x_{e,k}, y_{e,k})$ denotes the interpolated feed voltage amplitude at $(x_{e,k}, y_{e,k})$ and

$$s_{e,k}(n, m) = \sqrt{(x_{m} - x_{e,k})^{2} + (y_{n} - y_{e,k})^{2}}, \tag{7.10}$$
with \((x_{e,k}, y_{e,k})\) denoting the diffraction point of the Floquet wave for the \(k^{th}\) edge, which can be found from

\[
x_{e,k} = x_m - \frac{\beta_x |y_n + y_b|}{\sqrt{k_0^2 - \beta_x^2}}; |x_{e,k}| < x_b, y_{e,k} = \pm y_b \quad \text{for edges 2,4} \quad (7.11)
\]

\[
y_{e,k} = y_n - \frac{\beta_y |x_m + x_b|}{\sqrt{k_0^2 - \beta_y^2}}; |y_{e,k}| < y_b, x_{e,k} = \pm x_b \quad \text{for edges 1,3.} \quad (7.12)
\]

\(x_b, y_b\) in the above equations specify the boundary of the array as depicted in figure 7.2.

Finally, the function \(U_e\) is the Heaviside unit step function that defines the domain of existence of the edge diffraction. Its value is 1 when \((x_{e,k}, y_{e,k})\) lies on the physical edge, and 0 otherwise. The function \(f^{esw}\) associated with the edge-excited surface wave terms can be written as

\[
f^{esw}(n, m, k) = V(x'_{e,k}, y'_{e,k})e^{-j\beta_{sw}s'_{e,k}}e^{-j\beta_x x'_{e,k}}e^{-j\beta_y y'_{e,k}}U'_e(x'_{e,k}, y'_{e,k}), \quad (7.13)
\]

where \(V(x'_{e,k}, y'_{e,k})\) denotes the interpolated feed voltage amplitude at \((x'_{e,k}, y'_{e,k})\), \(\beta_{sw}\) denotes the surface wave propagation constant of the grounded medium, and

\[
s'_{e,k}(n, m) = \sqrt{(x_m - x'_{e,k})^2 + (y_n - y'_{e,k})^2}, \quad (7.14)
\]

with \((x'_{e,k}, y'_{e,k})\) denoting the point on the \(k^{th}\) edge where the edge-excited surface wave is launched, which can be found via

\[
x'_{e,k} = x_m - \frac{\beta_x |y_n + y_b|}{\sqrt{\beta_{sw}^2 - \beta_x^2}}; |x'_{e,k}| < x_b, y'_{e,k} = \pm y_b \quad \text{for edges 2,4} \quad (7.15)
\]

\[
y'_{e,k} = y_n - \frac{\beta_y |x_m + x_b|}{\sqrt{\beta_{sw}^2 - \beta_y^2}}; |y'_{e,k}| < y_b, x'_{e,k} = \pm x_b \quad \text{for edges 1,3.} \quad (7.16)
\]

The function \(U'_e\) is the Heaviside unit step function that defines the domain of existence of the edge-excited surface wave. Likewise, the function \(f^{csu,x}\) and \(f^{csu,y}\) associated with the
corner-excited surface wave terms can be written as

\[ f_{csw,x}(n, m, k) = V(x_{c,k}, y_{c,k})e^{-j\beta_{sw}s_{c,k}}e^{-j\beta_x x_{c,k}}e^{-j\beta_y y_{c,k}} \frac{k_{tx}}{\sqrt{2\pi \beta_{sw}s_{c,k}}} \beta_{sw} \cos \phi_c - \beta_x F(\xi_{cx})\sgn(x_{c,k}), \]

(7.17)

and

\[ f_{csw,y}(n, m, k) = V(x_{c,k}, y_{c,k})e^{-j\beta_{sw}s_{c,k}}e^{-j\beta_x x_{c,k}}e^{-j\beta_y y_{c,k}} \frac{k_{ty}}{\sqrt{2\pi \beta_{sw}s_{c,k}}} \beta_{sw} \sin \phi_{c,k} - \beta_y F(\xi_{cy})\sgn(y_{c,k}), \]

(7.18)

respectively, where \( V(x_{c,k}, y_{c,k}) \) denotes the interpolated feed voltage amplitude at \((x_{c,k}, y_{c,k})\) and

\[ s_{c,k}(n, m) = \sqrt{(x_m - x_{c,k})^2 + (y_n - y_{c,k})^2}, \]

(7.19)

with \((x_{c,k}, y_{c,k})\) denoting the coordinate of the \(k^{th}\) corner. \(\phi_{c,k}\) can be found from

\[ \phi_{c,k}(n, m) = \tan^{-1}\left( \frac{y_n - y_{c,k}}{x_m - x_{c,k}} \right), \]

(7.20)

where the range of \(\tan^{-1}(x)\) is \([-\pi, \pi]\), and \(\sgn(x)\) denotes the standard signum function.

Also, \(k_{tx}\) and \(k_{ty}\) are given by

\[ k_{tx} = \sqrt{\beta_{sw}^2 - \beta_x^2}, \]

(7.21)

and

\[ k_{ty} = \sqrt{\beta_{sw}^2 - \beta_y^2}, \]

(7.22)

respectively. Finally, \(F(x)\) denotes the Fresnel transition function given by

\[ F(x) = 2j\sqrt{x}e^{jx} \int_{\sqrt{x}}^{\infty} e^{-jt^2} dt, \quad -\frac{3\pi}{4} < \arg \sqrt{x} < \frac{\pi}{4}. \]

(7.23)
The arguments of the transition function $\xi_{cx}$ and $\xi_{cy}$ are given by

$$\xi_{cx} = 2\beta_{sw}s_{c,k}\sin^2\left(\frac{\phi_{c,k} - \phi_{cx}}{2}\right),$$

(7.24)

and

$$\xi_{cy} = 2\beta_{sw}s_{c,k}\sin^2\left(\frac{\phi_{c,k} - \phi_{cy}}{2}\right),$$

(7.25)

respectively, where $\phi_{cx}$ and $\phi_{cy}$ are given by

$$\phi_{cx} = \begin{cases} -\alpha_{sw} & \text{for corners 1, 2} \\ \alpha_{sw} & \text{for corners 3, 4} \end{cases},$$

(7.26)

$$\phi_{cx} = \begin{cases} \frac{\pi}{2} + \gamma_{sw} & \text{for corners 1, 4} \\ \frac{\pi}{2} - \gamma_{sw} & \text{for corners 2, 3} \end{cases},$$

(7.27)

with $\alpha_{sw} = \cos^{-1}(\beta_x/\beta_{sw})$ and $\gamma_{sw} = \cos^{-1}(\beta_y/\beta_{sw})$.

The unknowns for the edge parts can be expressed in a similar way as those for the inner part, but with much smaller number of array elements in each edge part and more number of unknowns associated with FW modal term is required, whereas the edge diffraction and edge-excited surface wave terms are almost the same as those for the inner part. First, each edge part that consists of either three rows or three columns is broken into three smaller parts which are now of only one row or column, and then an unknown coefficient is introduced to each smaller part to represent the FW modal term. Therefore, $i_{nm}^\text{edge}$ can be expressed as

$$i_{nm}^\text{edge,ij} \approx \sum_{k=1}^{4} \left[ \frac{j_{k,1}^{\text{ed}}}{\sqrt{s_{e,k}}} + \frac{j_{k,2}^{\text{ed}}}{\sqrt{s_{e,k}^3}} + \frac{j_{k,3}^{\text{ed}}}{\sqrt{s_{e,k}^5}} \right] f^{\text{ed}}(n, m, k) + \sum_{k=1}^{4} j_{k}^{\text{esw}} f^{\text{esw}}(n, m, k) + \sum_{k=1}^{4} [j_{k}^{\text{esw},x} f^{\text{esw},x}(n, m, k) + j_{k}^{\text{esw},y} f^{\text{esw},y}(n, m, k)],$$

(7.28)
where $j_{l,j}^{FW}$ is the unknown associated with the $j^{th}$ row/column of the $l^{th}$ edge, while other unknowns are the same as those of $j_{nm}^{inner}$. Since each edge part consists of either three rows or three columns, the number of unknowns associated with FW modal term is 3 for each edge part. It is noted that the $k^{th}$ edge diffraction is excluded for the elements inside the $k^{th}$ edge part and $j_k^{csw,x}$ and $j_k^{csw,y}$ are related to $j_k^{csw}$ as given by (7.4) and (7.5). All functions used in the above equation are the same as those used in (7.3). Using the newly-introduced UTD-based basis functions as described above, the new set of unknowns consists of

- Floquet modal wave: 1 unknown for the inner part, 3 unknowns for each edge part.
Figure 7.2: Region division for the UTD-MoM method. Also shown are, Floquet-modal wave, Floquet edge diffraction, edge excited surface wave, and corner excited surface wave.

- Floquet edge-diffracted wave: 3 unknowns for each edge part and are used for both inner and edge parts.

- Edge-excited surface wave: for each surface wave, 1 unknown for each edge part and is common for both inner array part and edge parts.

- Corner part: 15 unknowns for each corner.

Thus, the new total number of unknowns becomes $1 + 3 \times 4 + 3 \times 4 + 1 \times 4 + 15 \times 4 = 89$ per one expansion mode regardless of the number of array elements, when there is only one surface wave.
Now, define the vector $\mathbf{j}$ as

$$
\mathbf{j} = \begin{bmatrix}
    j^{iFW}_{J,1,1,1} & j^{ed}_{J,1,1,2} & \cdots & j^{ed}_{J,4,3,1} & \cdots & j^{esw}_{J,1,1,1} & \cdots & j^{esw}_{J,4,3,1} & j^{eFW}_{J,1,1,1} & \cdots & j^{eFW}_{J,4,3,1} & \cdots & j^{\text{corner}_{1,1}} & \cdots & j^{\text{corner}_{4,15}}
\end{bmatrix}^T, (7.29)
$$

where $i_{k,l}^{\text{corner}}$ denotes the unknown coefficient associated with the $l^{th}$ element of the $k^{th}$ corner, then for the single expansion mode case, the MoM current coefficient vector $\mathbf{i}$ can be given in terms of $\mathbf{j}$ as

$$
\mathbf{i} = \mathbf{Gj}, \quad (7.30)
$$

where $\mathbf{G}$ is a rectangular matrix relating $\mathbf{i}$ and $\mathbf{j}$ via the UTD-based basis functions. Substituting (7.30) into the MoM matrix equation yields

$$
\mathbf{ZGj} = \mathbf{v}, \quad (7.31)
$$

where $\mathbf{Z}$, $\mathbf{v}$ denote the MoM operator matrix and the excitation vector, respectively. Since the number of unknowns is reduced, it is evident that (7.31) becomes an overdetermined system and only the least squares solution can be obtained. To solve for the solution, the conventional normal equation approach can be applied by multiplying both sides of (7.31) by the Hermitian of $(\mathbf{ZG})$, i.e., $(\mathbf{ZG})^H$, but the computational cost can be expensive. Alternatively, it is found in this work that using $\mathbf{G}^H$ instead of $(\mathbf{ZG})^H$ can also lead to a reasonably accurate solution. It is noted that the matrix $\mathbf{G}$ is highly sparse and $\mathbf{Z}$ is a BTTB matrix, thus the computational cost of the matrix-matrix product $\mathbf{ZG}$ is not that significant. Using this approach, the least squares solution to this problem can be given by

$$
\mathbf{j} = (\mathbf{G}^H \mathbf{ZG})^{-1} \mathbf{G}^H \mathbf{v}. \quad (7.32)
$$

Since the dimension of $\mathbf{G}^H \mathbf{ZG}$ is equal to the new total number of unknowns, which is not large, a direct matrix solver can be applied, and it helps eliminate the convergence problem due to a large condition number. Once $\mathbf{j}$ is obtained, $\mathbf{i}$ can be found from (7.30).
This approach can be easily extended to the case of multiple expansion modes by representing the unknowns for each expansion mode in terms of the corresponding new unknowns of the same mode via the UTD-based basis functions as

\[ i_k = G_j, \quad (7.33) \]

where \( i_k \) is the coefficient vector for the \( k^{th} \) expansion mode and \( j \) is the corresponding new unknown vector for the \( k^{th} \) mode. Now, introducing a new matrix \( G_{N_{mode}} \)

\[ G_{N_{mode}} = \hat{I}_{N_{mode}} \otimes G, \quad (7.34) \]

where \( N_{mode} \) is the number of expansion modes, \( \hat{I}_{N_{mode}} \) denotes the \( N_{mode} \times N_{mode} \) identity matrix, and the operator \( \otimes \) in the above equation denotes the Kronecker product, then the MoM unknown coefficient vector for the case of multiple expansion modes can be given by

\[ i = G_{N_{mode}} j, \quad (7.35) \]

where \( i = [i_1^T \cdots i_{N_{mode}}^T]^T \) and \( j = [j_1^T \cdots j_{N_{mode}}^T]^T \). It follows that the least squares system for this case is given by

\[ (G_{N_{mode}}^H Z G_{N_{mode}}) j = G_{N_{mode}}^H v, \quad (7.36) \]

where \( v = [v_1^T \cdots v_{N_{mode}}^T]^T \), and

\[ G_{N_{mode}}^H Z G_{N_{mode}} = \begin{bmatrix}
G^H & G^H & \cdots & G^H \\
G^H & G^H & \cdots & G^H \\
\vdots & \vdots & \ddots & \vdots \\
G^H & G^H & \cdots & G^H \\
G^H Z_1 & G^H Z_1 & \cdots & G^H Z_{N_{mode}} \\
G^H Z_2 & G^H Z_2 & \cdots & G^H Z_{N_{mode}} \\
\vdots & \vdots & \ddots & \vdots \\
G^H Z_{N_{mode}} & G^H Z_{N_{mode}} & \cdots & G^H Z_{N_{mode}}
\end{bmatrix}
\]

\[ = \begin{bmatrix}
Z_1 & Z_1 & \cdots & Z_{N_{mode}} \\
Z_2 & Z_2 & \cdots & Z_{N_{mode}} \\
\vdots & \vdots & \ddots & \vdots \\
Z_{N_{mode}} & Z_{N_{mode}} & \cdots & Z_{N_{mode}}
\end{bmatrix}
\]

\[ = \begin{bmatrix}
G^H & G^H & \cdots & G^H \\
G^H & G^H & \cdots & G^H \\
\vdots & \vdots & \ddots & \vdots \\
G^H & G^H & \cdots & G^H \\
G^H Z_1 & G^H Z_1 & \cdots & G^H Z_{N_{mode}} \\
G^H Z_2 & G^H Z_2 & \cdots & G^H Z_{N_{mode}} \\
\vdots & \vdots & \ddots & \vdots \\
G^H Z_{N_{mode}} & G^H Z_{N_{mode}} & \cdots & G^H Z_{N_{mode}}
\end{bmatrix}
\]

\[ = \begin{bmatrix}
G^H & G^H & \cdots & G^H \\
G^H & G^H & \cdots & G^H \\
\vdots & \vdots & \ddots & \vdots \\
G^H & G^H & \cdots & G^H \\
G^H Z_1 & G^H Z_1 & \cdots & G^H Z_{N_{mode}} \\
G^H Z_2 & G^H Z_2 & \cdots & G^H Z_{N_{mode}} \\
\vdots & \vdots & \ddots & \vdots \\
G^H Z_{N_{mode}} & G^H Z_{N_{mode}} & \cdots & G^H Z_{N_{mode}}
\end{bmatrix}
\]

\[ = \begin{bmatrix}
G^H & G^H & \cdots & G^H \\
G^H & G^H & \cdots & G^H \\
\vdots & \vdots & \ddots & \vdots \\
G^H & G^H & \cdots & G^H \\
G^H Z_1 & G^H Z_1 & \cdots & G^H Z_{N_{mode}} \\
G^H Z_2 & G^H Z_2 & \cdots & G^H Z_{N_{mode}} \\
\vdots & \vdots & \ddots & \vdots \\
G^H Z_{N_{mode}} & G^H Z_{N_{mode}} & \cdots & G^H Z_{N_{mode}}
\end{bmatrix}
\]

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7.3 An Extension to Arrays with Non-rectangular Element Truncation Boundaries

As mentioned in the previous chapter, practical arrays may have non-rectangular element truncation boundaries, thus it is desirable to extend the hybrid approach developed here to handle these non-rectangular arrays as well. The difficulty here is that non-rectangular boundaries may not necessarily be formed by exact straight lines as those of rectangular arrays. However, by approximating them using only straight edges, the approach used for rectangular arrays can also be applied. As discussed in the previous section, the array field consists of the Floquet modal waves and the diffraction and surface wave components produced at the array boundary. Thus, the behavior of unknown current coefficients, denoted here by $i_{nm}$, in the array interior can be approximated by the infinite array solution, together with the modifications resulting from the effects of the finiteness of the arrays, as in the case of rectangular arrays. The only difference from the case of rectangular arrays is that the boundary consists of more than four edges and some edges may not align with the $\hat{x}$ or $\hat{y}$ axis. For an octagonal array shown in figure 7.3, it can be divided into the outer part, which consists of eight edges and eight corners, and the remaining inner part. Each edge part is confined to a distance of three cells from the actual edge of the array, as in the rectangular array case, whereas each corner part is confined to a region filled by nine cells. Using the region division mentioned above, the functional variation of $i_{nm}$ for a non-rectangular array using a global, UTD-based approximation, can be given in the same way as that for a rectangular array, i.e., that given by (7.2).
Figure 7.3: Region division for the hybrid UTD-MoM method in the case of octagonal arrays

Using the octagonal array shown in figure 7.3 as an example, $\imath_{\text{inner}}^{nm}$ can be expressed as

$$
\imath_{\text{inner}}^{nm} \approx j^{iFW} f^{FW}(n, m) + \sum_{k=1}^{8} \left[ \frac{j_{k,1}^{ed}}{\sqrt{s_{e,k}}} + \frac{j_{k,2}^{ed}}{\sqrt{s_{e,k}^3}} + \frac{j_{k,3}^{ed}}{\sqrt{s_{e,k}^5}} \right] f^{ed}(n, m, k)
+ \sum_{k=1}^{8} j_{k}^{esw} f^{esw}(n, m, k),
$$

$$
+ \sum_{k=1}^{8} \left[ j_{k}^{esw,\eta} f^{esw,\eta}(n, m, k) + j_{k}^{esw,\zeta} f^{esw,\zeta}(n, m, k) \right],
$$

(7.38)

where $j^{iFW}$, $j_{k,l}^{ed}$, $k \in \{1, 2, 3, \ldots, 8\}$, $l \in \{1, 2, 3\}$, and $j_{k}^{esw}, k \in \{1, 2, 3, \ldots, 8\}$ are the new unknown coefficients. As in the case of rectangular arrays, $f^{esw,\eta}$ and $f^{esw,\zeta}$ are
introduced to compensate the discontinuities of the corresponding edge-excited surface wave in the shadow regions, thus \( \{ j_{csw,\eta}^1, \cdots, j_{csw,\eta}^8 \} \) and \( \{ j_{csw,\zeta}^1, \cdots, j_{csw,\zeta}^8 \} \) are related to \( \{ j_{esw}^1, \cdots, j_{esw}^8 \} \). For instance, \( j_{csw,\eta}^1 = j_{esw}^7 \) and \( j_{csw,\zeta}^1 = j_{esw}^8 \) and vice versa.

The functions \( f_{FW}, f_{ed}, f_{esw} \) in (7.38) are the same functional forms given in (7.6), (7.9), and (7.13), respectively. The diffraction points for edges 1, 3, 5 and 7, which align with \( \hat{x} \) axis or \( \hat{y} \) axis, can be found in the same way as before, whereas those for other edges can be obtained from finding the intersection points of edges and the edge-diffracted rays.

Let \( \hat{e} \) denote the normal vector representing the direction of the edge, i.e.,

\[
\hat{e} = \hat{x}e_x + \hat{y}e_y = \frac{1}{\sqrt{1 + e_{yx}^2}}(\hat{x} + \hat{y}e_{yx}),
\]

(7.39)

with \( e_{yx} = e_y/e_x \), the direction of the edge-diffracted rays can be given by

\[
\hat{s} = \frac{1}{\sqrt{1 + \alpha^2_{\pm}}}(\hat{x} + \hat{y}\alpha_{\pm}),
\]

(7.40)

where the plus sign denotes the case where \( \hat{s} \) is rotated counter-clockwise from \( \hat{e} \) and the minus sign denotes the clockwise rotation case. \( \alpha_{\pm} \) can be found from

\[
\alpha_{\pm} = \frac{\pm \sin \beta + e_{yx}\cos \beta}{\cos \beta \mp e_{yx}\sin \beta},
\]

(7.41)

where \( \beta \) denotes the angle between the edge and the FW modal wave given by

\[
\cos \beta = e_x\beta_x + e_y\beta_y.
\]

(7.42)

Now, let \((x_1, y_1)\) be a point on the edge, then the diffraction points can be given by

\[
x_e = \frac{y_n - y_1 + e_{yx}x_1 - \alpha_{\pm}x_m}{e_{yx} - \alpha_{\pm}}, \quad y_e = \alpha_{\pm}(x_e - x_m) + y_n.
\]

(7.43)
Likewise, using the same procedure, the launching points of edge-excited surface waves, 
\((x'_e, y'_e)\), can be obtained as well.

Using the same approach used for rectangular arrays, \(i_{nm}^{\text{edge}}\) can be expressed as

\[
\begin{align*}
\bar{i}_{nm}^{\text{edge},lj} & \approx j_{i,j}^{\text{eFW}} f^{\text{FW}}(n,m) + \sum_{k=1, k \neq l}^{8} \left[ \frac{j_{k,1}^{\text{ed}}}{\sqrt{s_{e,k}}} + \frac{j_{k,2}^{\text{ed}}}{\sqrt{s_{e,k}^3}} + \frac{j_{k,3}^{\text{ed}}}{\sqrt{s_{e,k}^5}} \right] f^{\text{ed}}(n,m,k) \\
& + \sum_{k=1}^{8} j_{k}^{\text{esw}} f^{\text{esw}}(n,m,k) \\
& + \sum_{k=1}^{8} \left[ j_{k}^{\text{esw,\eta}} f^{\text{esw,\eta}}(n,m,k) + j_{k}^{\text{esw,\zeta}} f^{\text{esw,\zeta}}(n,m,k) \right],
\end{align*}
\]

(7.45)

where \(j_{i,j}^{\text{eFW}}\) is the unknown associated with the \(j^{\text{th}}\) row/column of the \(l^{\text{th}}\) edge, as before.

All functions used in the above equation are the same as those used in (7.38). Using the newly-introduced UTD-based basis functions as described above, the new set of unknowns consists of

- Floquet modal wave: 1 unknown for the inner part, 3 unknowns for each edge part.
- Floquet edge-diffracted wave: 3 unknowns for each edge part and are used for both inner and edge parts.
- Edge-excited surface wave: for each surface wave, 1 unknown for each edge and is common for both inner array part and edge parts.
- Corner part: 9 unknowns for each corner.

Thus, the new total number of unknowns becomes \(1 + 3 \times 8 + 3 \times 8 + 1 \times 8 + 9 \times 8 = 129\) per one expansion mode for an octagonal array, when there is only one surface wave.

Now, define the vector \(\vec{j}\) as

\[
\vec{j} = \begin{bmatrix}
  j_{1,1}^{\text{FW}} & j_{1,2}^{\text{FW}} & \cdots & j_{8,3}^{\text{FW}} & j_{8}^{\text{esw}} & j_{8}^{\text{FW}} & \cdots & j_{8,3}^{\text{FW}} & i_{1,1}^{\text{corner}} & \cdots & i_{8,9}^{\text{corner}}
\end{bmatrix}^T,
\]

(7.46)
where \( i_{\text{corner}}^{k,l} \) denotes the unknown coefficient associated with the \( l^{th} \) element of the \( k^{th} \) corner, then for the single expansion mode case, the MoM current coefficient vector \( \vec{i} \) can be given in terms of \( \vec{j} \) as

\[
\vec{i} = G \vec{j},
\]

(7.47)

where \( G \) is a rectangular matrix relating \( \vec{i} \) and \( \vec{j} \) via the UTD-based basis functions. It is noted that \( G \) here is a \( K \times 129 \) matrix, where \( K \) denotes the total number of elements in the octagonal array. Substituting (7.30) into the MoM matrix equation yields

\[
Z_A G \vec{j} = \vec{v},
\]

(7.48)

where \( Z_A, \vec{v} \) denote the MoM operator matrix and the excitation vector, respectively. As discussed in the previous chapter, the MoM operator matrix for an octagonal array can be obtained from that for the rectangular array which circumscribes the octagonal array by

\[
Z_A = A^T Z A,
\]

(7.49)

where \( A \) denotes the proper array shape matrix that is discussed in chapter 6. Therefore, (7.48) becomes

\[
(A^T Z A) G \vec{j} = \vec{v}.
\]

(7.50)

Using the same procedure used for the rectangular array case, the least squares solution to this problem can be given by

\[
\vec{j} = (G^H A^T Z A G)^{-1} G^H \vec{v}.
\]

(7.51)

Now, let \( G_A = A G \), then \( G_A \) is an \( NM \times 129 \) matrix, where \( M \times N \) denotes the size of the rectangular array that encloses the octagonal array. Using \( G_A \), the above equation can be simplified to be

\[
\vec{j} = (G_A^H Z G_A)^{-1} G_A^H \vec{v}.
\]

(7.52)
Likewise, the least squares system for the case of multiple expansion modes can be given by

\[
(G^H_{A,N_mode} Z G_{A,N_mode}) \mathbf{j} = G^H_{N_mode} \mathbf{v},
\]

(7.53)

where

\[
\mathbf{j} = [j^T \cdots \mathbf{j}_{N_mode}^T]^T, \quad \mathbf{v} = [v^T \cdots v_{N_mode}^T]^T,
\]

\[
G_{A,N_mode} = A_{N_mode} G_{N_mode}
\]

\[
= (\hat{I}_{N_mode} \otimes A)(\hat{I}_{N_mode} \otimes G)
\]

\[
= \hat{I}_{N_mode} \otimes (A G)
\]

\[
= \hat{I}_{N_mode} \otimes G_A,
\]

and

\[
G^H_{A,N_mode} Z_A G_{A,N_mode}
\]

\[
= \begin{bmatrix}
G^H_A & G^H_A & \cdots & G^H_A \\
\vdots & \vdots & \ddots & \vdots \\
G^H_A & G^H_A & \cdots & G^H_A \\
\end{bmatrix}
\begin{bmatrix}
Z^{1,1} & Z^{1,2} & \cdots & Z^{1,N_mode} \\
Z^{2,1} & Z^{2,2} & \cdots & Z^{2,N_mode} \\
\vdots & \vdots & \ddots & \vdots \\
Z^{N_mode,1} & Z^{N_mode,2} & \cdots & Z^{N_mode,N_mode} \\
\end{bmatrix}
\begin{bmatrix}
G_A \\
G_A \\
\vdots \\
G_A \\
\end{bmatrix}
\]

(7.55)

7.4 Numerical Results

Several numerical results are shown here to demonstrate the effectiveness of the hybrid UTD-MoM method developed in this work. First, results regarding arrays of various types of elements with rectangular element truncation boundaries are presented in section 7.4.1 followed by those of arrays with non-rectangular boundaries in section 7.4.2.
7.4.1 Rectangular Arrays

In this subsection, all arrays are assumed to have rectangular element truncation boundaries, i.e., a $M \times N$ array means an array consisting of $N$ rows with $M$ elements in each row. The first example deals with the radiation from an array of printed dipoles in multilayered medium. The dimension of printed dipoles, the element spacings as well as the grounded multilayered medium used in the calculation are shown in figure 7.4, where the unit of length is cm. The number of expansion modes per element is 3. Figure 7.5 shows the E-plane radiation pattern of the $128 \times 128$ array of these printed dipoles, when scanned at $(\theta^s, \phi^s) = (30^\circ, 0^\circ)$ and operating at 10 GHz. The plot shows patterns obtained by both hybrid UTD-MoM method and conventional MoM using preconditioned iterative solvers also developed in this work. As can be seen, excellent agreement is observed between results obtained by two methods except for the extremely far out sidelobes.

![Printed dipole array and multilayered medium](image)

**Figure 7.4:** Geometry of the array of printed dipoles used in the calculations

As the second example, radiation/scattering from an array of microstrip-line fed patch antennas on single-layer substrate are analyzed. Figure 7.6 shows the E-plane radiation pattern of a $128 \times 128$ microstrip-line fed patch antenna array on a single-layered substrate.
Figure 7.5: E-plane radiation pattern of a $128 \times 128$ printed dipole array for $(\theta^i, \phi^i) = (30^\circ, 0^\circ)$.

The dimension of the patch antenna used here is $4.02 \times 4.02$ cm. and the width of the feed line is $0.445$ cm. The spacings in both directions are $5$ cm. and the operating frequency is $2.28$ GHz. The substrate is $0.159$ cm. thick and its dielectric constant is $2.55 - j0.0051$. The plot shows patterns obtained by both hybrid UTD-MoM and conventional MoM methods; as can be seen, good agreement is observed here as well. Likewise, both TE and TM bistatic RCS patterns obtained by two methods of two $128 \times 128$ patch antenna arrays are shown in figure 7.7, and 7.8, respectively, from which a good agreement can be seen as well. The parameters used in the calculations here are given in table 7.1. The incident angle for both arrays is $(\theta^i, \phi^i) = (30^\circ, 0^\circ)$. It can be seen from figure 7.8 that this array exhibits a grating lobe at around $76^\circ$ in the E-plane, which demonstrates the effectiveness of this hybrid method even when there exists a grating lobe.
Table 7.1: Parameters of two patch antenna arrays used in the calculations

<table>
<thead>
<tr>
<th>Parameter</th>
<th>First Array</th>
<th>Second Array</th>
</tr>
</thead>
<tbody>
<tr>
<td>Element Dimension $(L, W)$</td>
<td>(1.008, 1.179) cm.</td>
<td>(3.66, 2.6) cm.</td>
</tr>
<tr>
<td>Element Spacings $(d_x, d_y)$</td>
<td>(1.232, 1.232) cm.</td>
<td>(5.5517, 5.5517) cm.</td>
</tr>
<tr>
<td>Substrate Dielectric Constant</td>
<td>$2.55 - j0.0025$</td>
<td>2.17</td>
</tr>
<tr>
<td>Substrate Thickness</td>
<td>0.158 cm.</td>
<td>0.158 cm.</td>
</tr>
<tr>
<td>Frequency</td>
<td>9.42 GHz.</td>
<td>3.7 GHz.</td>
</tr>
</tbody>
</table>

The next example deals with radiation/scattering from a $128 \times 128$ probe-fed patch antenna array on a single-layered substrate. The dimension of the patch antenna is $(L, W) = (2.0, 3.0)$ cm. and the spacings in both directions are 4.0 cm. The probes are located 0.65 cm. from the edge of each patch and its inner, outer radii are 0.043, 0.14, cm. respectively. The substrate is 0.127 cm. thick and its dielectric constant is $10.2 - j0.051$. Figure 7.9 shows the radiation pattern when scanned at $(\theta^*, \phi^*) = (30^\circ, 0^\circ)$ and operating at 2.3025 GHz., whereas figure 7.10 shows both TE and TM bistatic RCS patterns of this probe-fed patch antenna array at the same frequency. As before, a good agreement can be observed here as well.
Figure 7.6: E-plane radiation pattern of a 128 \times 128 microstrip-line fed patch antenna array for \((\theta^i, \phi^i) = (30^\circ, 0^\circ)\).

Figure 7.7: TE and TM bistatic RCS patterns of the first 128 \times 128 patch antenna array for \((\theta^i, \phi^i) = (30^\circ, 0^\circ)\).
Figure 7.8: TE and TM bistatic RCS patterns of the second $128 \times 128$ patch antenna array for $(\theta^i, \phi^i) = (30^\circ, 0^\circ)$ when there exists a grating lobe.

Figure 7.9: E-plane radiation pattern of a $128 \times 128$ probe-fed patch antenna array for $(\theta^i, \phi^i) = (30^\circ, 0^\circ)$. 
Figure 7.10: TE and TM bistatic RCS patterns of a 128 × 128 probe-fed patch antenna array for \((\theta^i, \phi^i) = (30^\circ, 0^\circ)\).
7.4.2 Non-rectangular Arrays

In this subsection, several numerical results regarding arrays with octagonal element truncation boundary shown in figure 7.11 are presented. The total number of this array is 7,592, and it is enclosed by a $96 \times 96$ square array, whose number of elements is 9,216. The geometries of all arrays as well as the frequencies are the same as used in the cases of rectangular array, except the element truncation boundaries are changed to octagonal ones.

![Octagonal element truncation boundary used in the calculations.](image)

Figure 7.11: Octagonal element truncation boundary used in the calculations.

Figure 7.12 shows the E-plane radiation pattern of the printed dipole array with octagonal element truncation boundary when scanned at $(\theta^s, \phi^s) = (30^\circ, 0^\circ)$. As before, the plot shows results obtained by both the hybrid UTD-MoM and conventional MoM methods. A good agreement between the two methods can be observed here as well. Likewise, figure 7.13 shows the radiation pattern of octagonal microstrip-line fed patch antenna array on
a single-layered substrate, while both TE and TM bistatic RCS patterns of two octagonal patch antenna arrays are shown in figure 7.14, and 7.15, respectively. All of them demonstrate good agreements between two methods and it is noted that a grating lobe is observed in the bistatic RCS patterns of the second patch array as in the rectangular array case.

![E-plane radiation pattern: Octagonal printed dipole array](image)

**Figure 7.12:** E-plane radiation pattern of an octagonal printed dipole array for \((\theta^i, \phi^i) = (30^\circ, 0^\circ)\).

Finally, Figure 7.16 shows the radiation pattern of the octagonal probe-fed patch antenna array when scanned at \((\theta^s, \phi^s) = (30^\circ, 0^\circ)\) and operating at 2.3025 GHz., whereas figure 7.17 shows both TE and TM bistatic RCS patterns of this octagonal probe-fed patch antenna array at the same frequency. As before, a good agreement can be observed here as well.
E-plane radiation pattern: octagonal patch antenna array

Figure 7.13: E-plane radiation pattern of an octagonal microstrip-line fed patch antenna array for \((\theta^s, \phi^s) = (30^\circ, 0^\circ)\).

(a) TE bistatic RCS
(b) TM bistatic RCS

Figure 7.14: TE and TM bistatic RCS patterns of the first patch antenna array with octagonal element truncation boundary for \((\theta^i, \phi^i) = (30^\circ, 0^\circ)\).
Figure 7.15: TE and TM bistatic RCS patterns of the second patch antenna array with octagonal element truncation boundary for \((\theta^i, \phi^i) = (30^\circ, 0^\circ)\) when there exists a grating lobe.

Figure 7.16: E-plane radiation pattern of an octagonal probe-fed patch antenna array for \((\theta^s, \phi^s) = (30^\circ, 0^\circ)\).
TE bistatic RCS pattern: octagonal probe-fed patch antenna array, $\theta_i=30^\circ$, $\phi_i=0^\circ$

TM bistatic RCS pattern: octagonal probe-fed patch antenna array, $\theta_i=30^\circ$, $\phi_i=0^\circ$

Figure 7.17: TE and TM bistatic RCS patterns of an octagonal probe-fed patch antenna array for $(\theta^i, \phi^i) = (30^\circ, 0^\circ)$. 
CHAPTER 8

CONCLUSIONS AND FUTURE WORK

8.1 Conclusions

A fast MoM-based full-wave solver, named here as the PI-MoM, has been developed in this work to predict radiation/scattering from large planar finite periodic arrays of various printed elements in grounded multilayered media. An efficient method to evaluate the multilayered media Green’s functions is implemented in this solver to reduce the computation time for filling the MoM operator matrix and the excitation vector. This method consists of an accurate closed-form asymptotic approximation for moderate to large source-observation point separations, and an efficient numerical integration method for small separations. The asymptotic closed-form approximation is considerably useful for large electromagnetic problems, since the results consist of only few terms. Several iterative solvers as well as a DFT-based preconditioner have also been implemented to reduce the storage and solving time required for solving the MoM matrix equation. Also included in this solver is the capability to handle arrays with non-rectangular element truncation boundaries without sacrificing any efficiency.
A completely separate hybrid UTD-MoM method has also been developed in this work to be used as an alternative matrix solver, especially when the convergence rate of the PI-MoM in some special cases worsens due to the increase in the condition number. Insofar as the radiation pattern or RCS pattern is concerned, this hybrid method has demonstrated that it can provide a good agreement with the PI-MoM. Its advantages include the fixed number of unknowns regardless of number of elements and the use of direct matrix solver, which eliminates the convergence problem. In fact, it is found to be much more efficient than the conventional MoM when an iterative solver requires many iterations to converge to a solution. Since it can be incorporated into the conventional MoM method without changing MoM basis functions, it has a potential to be applicable to any large array problem.

Overall, the development of the PI-MoM and the hybrid UTD-MoM approaches, respectively, provides some reasonable techniques for solving electromagnetic problems involving large finite arrays in planar grounded multilayered media. Although the individual types of printed elements studied in this work are limited to those with rectangular shapes, the solver developed here is expected to be easily extended to analyze arrays of more complicated elements as well as reflectarrays and frequency selective surfaces (FSS). It is also noted that the multilayered media Green’s functions obtained in this study are sufficient to formulate an integral equation for any antenna element type in a grounded multilayered medium.

### 8.2 Future Work

Although the PI-MoM based full-wave solver developed in this work can efficiently predict the electromagnetic radiation/scattering from large finite planar periodic arrays of various element types in grounded multilayered media to a certain level, it would definitely
benefit from some further enhancements to improve the efficiency and range of applications. The first issue should be addressed to the convergence rate of iterative solvers. As mentioned earlier, the convergence rate generally becomes worse as the number of unknowns increases, which is due to the increase in the condition number of the MoM operator matrix. Although a DFT-based preconditioner implemented in this full-wave solver can significantly help improve the convergence rate, it obviously has certain limitations on the array size, especially when the medium is quite thick and its dielectric constant is large. This problem can be alleviated by introducing a more effective preconditioner such as the one in [48]. However, such preconditioners would typically require additional memory storage and larger computational cost to implement, which potentially makes the solver less efficient. Since the hybrid UTD-MoM method, also developed here, is not sensitive to the condition number of the MoM operator matrix, and its computational cost does not significantly increases as the number of elements increases, it can therefore be used as an alternative solver, and thus improving it can lead to more accuracy and efficiency.

Presently, the solver can handle only element types with rectangular shapes, which is primarily due to the piecewise sinusoidal expansion functions used in the MoM implementation. By implementing more flexible basis functions, such as the Rao-Wilton-Glisson (RWG) triangular rooftop function, and using an additional CAD file to store all mesh information, the solver can be easily extended to handle printed antenna elements with arbitrary shapes as well. This would significantly broaden the range of the solver, since it can be used to analyze finite frequency selective surface (FSS) or finite arrays of antenna with more complicated shape, such as Vivaldi antenna.

Lastly, this solver can handle only arrays in grounded multilayered media. However, finite arrays in some practical applications, such as the band-stop FSS, may not have a
ground plane. Thus, it is also desirable to modify the solver such that it can analyze arrays in multilayered media but without a ground plane. The approach used in this work to evaluate the multilayered media dyadic Green’s functions is expected to be applicable to those without a ground plane as well, therefore the modification should require only a minimal effort.
APPENDIX A

DETAILS OF NUMERICAL INTEGRATION METHOD

In this appendix, details of numerical integration methods used to evaluate each function associated with the multilayered Green’s function described previously in chapter 2 will be given. The key point here is to obtain a good large-argument approximation, which can also be integrated analytically when substituting it in the integrand. It is also noted that this large-argument term is equivalent to the quasi-static term in the DCIM method.

A.1 Calculation of $U^{HE}$ and $W^{HE}$

As discussed previously in section 3.1, the large argument approximation of $f(\xi_1, \cdots, \xi_n)g(\lambda)$ in (3.1), denoted by $f^\infty(\lambda)$, has to be found to approximate the integrand for the range where $\lambda$ is large. For large $\lambda$, the following approximations can be made.

$$\xi_i = -j\lambda \quad \text{for } \forall i,$$

where $i = 1, \cdots, n$.

$$D^m_i = \left[ \begin{array}{c} \cos(\xi_{i-1}\Delta z_{i-1}) \quad j\frac{k_0}{\xi_{i-1}} \sin(\xi_{i-1}\Delta z_{i-1}) \\ j\frac{\epsilon_i}{\epsilon_{i-1}k_0} \sin(\xi_{i-1}\Delta z_{i-1}) \quad \frac{\epsilon_i}{\epsilon_{i-1}} \cos(\xi_{i-1}\Delta z_{i-1}) \end{array} \right]$$

$$= \frac{e^{j\xi_{i-1}\Delta z_{i-1}}}{2} \left[ \begin{array}{c} 1 + e^{-j2\xi_{i-1}\Delta z_{i-1}} \\ \frac{\epsilon_{i-1}}{\epsilon_i} \frac{\epsilon_i}{\epsilon_{i-1}} \frac{k_0}{\xi_{i-1}} (1 - e^{-j2\xi_{i-1}\Delta z_{i-1}}) \end{array} \right]$$

$$\approx \frac{e^{\lambda\Delta z_{i-1}}}{2} \left[ \begin{array}{c} 1 - \frac{k_0}{\epsilon_i} \\ \frac{\epsilon_i}{\epsilon_{i-1}k_0} \frac{\epsilon_i}{\epsilon_{i-1}} \frac{\lambda}{\epsilon_{i-1}} \end{array} \right],$$

(A.2)
\[
\mathcal{D}_i = \begin{bmatrix}
cos(\xi_{i-1} \Delta z_{i-1}) & j \frac{\mu_i k_0}{\mu_{i-1} k_0} \sin(\xi_{i-1} \Delta z_{i-1}) \\
j \frac{\mu_i k_0}{\mu_{i-1} k_0} \sin(\xi_{i-1} \Delta z_{i-1}) & \frac{\mu_i}{\mu_{i-1}} \cos(\xi_{i-1} \Delta z_{i-1})
\end{bmatrix}
\]

Using the above results, one obtains \( \mathcal{U}_{i}^m, \mathcal{U}_{i}^{m,HE} \) for large \( \lambda \) as

\[
\mathcal{D}_{i}^{m,HE} = \begin{cases}
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & m = 2, \\
2 \prod_{i=m-1}^{\infty} \mathcal{D}_i & m = n,
\end{cases}
\]

\[
\mathcal{U}_{i}^{m,HE} = \begin{cases}
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & m = 2, \\
2 \prod_{i=m}^{\infty} \mathcal{U}_i & m = n,
\end{cases}
\]

where

\[
\mathcal{D}_{11}^{\infty} = \begin{cases}
\frac{e^{\lambda \Delta z_{m-1}}}{2} & m = 3 \\
\prod_{i=m-2}^{\infty} (1 + \frac{\epsilon_i}{\epsilon_{i-1}}) & m = n - 1
\end{cases}
\]

\[
\mathcal{U}_{11}^{\infty} = \begin{cases}
\frac{e^{\lambda \Delta z_{m-1}}}{2} & m = 3 \\
\prod_{i=m}^{\infty} (1 + \frac{\epsilon_i}{\epsilon_{i-1}}) & m = n - 1
\end{cases}
\]
Likewise, $\mathcal{D}^{e,HE}, U^{e,HE}$ for large $\lambda$ can be found to be

$$
\mathcal{D}^{e,HE} = \begin{cases}
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; & m = 2, \\
\prod_{i=m-1}^{2} D_i^e \simeq D_{11}^{\infty} \begin{bmatrix} 1 & -\frac{\mu_{m-1} \lambda}{\mu_{m-2} k_0} \\ j \frac{\lambda}{\mu_{m-1}} & -\frac{k_0}{\mu_{m-2}} \end{bmatrix}; & \text{otherwise},
\end{cases}
$$
(A.10)

$$
U^{e,HE} = \begin{cases}
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; & m = n, \\
\prod_{i=m}^{n-1} U_i^m \simeq U_{11}^{\infty} \begin{bmatrix} 1 & \frac{\mu_{n-1} k_0}{j \frac{\lambda}{k_0}} \mu_n \\ \frac{j \frac{\lambda}{k_0}}{\mu_{n-1}} & \frac{\mu_n}{\mu_n} \end{bmatrix}; & \text{otherwise}.
\end{cases}
$$
(A.11)

Now, using the above approximations to approximate $f(\xi_1, \ldots, \xi_n)$ for $U^{HE}$ yields

$$
f^{\infty}_{U^{HE}} = \left. \frac{C^{e',HE}_m}{k_0^2 \Delta e,HE} \right|_{\lambda \to \infty} = -\frac{j}{2k_0 \lambda} \frac{\mu_{m-1}}{\mu_{av}},
$$
(A.12)

where $\mu_{av} = (\mu_m + \mu_{m-1})/2$. Using $\int_{\lambda_{w}}^{\infty} \{\bullet\} = \int_{0}^{\infty} \{\bullet\} - \int_{0}^{\lambda_{w}} \{\bullet\}$ and the following identity

$$
\int_{0}^{\infty} J_0(\alpha x) dx = \frac{1}{\alpha},
$$
(A.13)

one obtains

$$
\int_{0}^{\infty} f^{\infty}_{U^{HE}} J_0(\lambda) \lambda d\lambda = -\frac{j}{2k_0 \rho} \frac{\mu_{m-1}}{\mu_{av}}.
$$
(A.14)

Therefore, $U^{HE}$ can be given by

$$
U^{HE} = \left\{ -\frac{j}{2k_0 \rho} + \frac{1}{k_0^2} \int_{0}^{\lambda_{m}} \left( \frac{C^{e',HE}_m}{\Delta e,HE} \lambda + \frac{jk_0}{2} \right) J_0(\lambda \rho) d\lambda \right\} \frac{\mu_{m-1}}{\mu_{av}}.
$$
(A.15)

Applying the same approach to $W^{HE}$ yields

$$
f^{\infty}_{W^{HE}} = \left. \left( \frac{C^{e',HE}_m}{\Delta e,HE} - \frac{k_0^2}{k_0^2 \mu_{m} \Delta m,HE} \right) \right|_{\lambda \to \infty}
$$

$$
= \left[ -\frac{j}{2k_0 \lambda} - \frac{j \lambda}{2k_0 \epsilon_{av}} + \frac{jk_0}{8} \frac{(\epsilon_{m} - \epsilon_{m-1})^2}{\epsilon_{av}^2} \right] \frac{\mu_{m-1}}{\mu_{av}}
$$

$$
= \left[ -\frac{j}{2k_0 \lambda} - \frac{j \lambda}{2k_0 \epsilon_{av}} \right] \frac{\mu_{m-1}}{\mu_{av}}.
$$
(A.16)
where $\epsilon_{av} = (\epsilon_m + \epsilon_{m-1})/2$ and $\alpha = 1 - \frac{(\epsilon_m - \epsilon_{m-1})^2}{4\epsilon_{av}^2}$. Since

$$\int_{\lambda_m}^{\infty} -\frac{j\lambda}{2k_0 \epsilon_{av}} \frac{J_0(\lambda \rho)}{\lambda} d\lambda = \frac{j}{2k_0 \epsilon_{av}} \left[ -\int_{0}^{\infty} J_0(\lambda \rho) d\lambda + \int_{0}^{\lambda_m} J_0(\lambda \rho) d\lambda \right]$$

$$= \frac{j}{2k_0 \epsilon_{av}} \left[ -\frac{1}{\rho} + \int_{0}^{\lambda_m} J_0(\lambda \rho) d\lambda \right], \quad (A.17)$$

and

$$\int_{\lambda_m}^{\infty} -\frac{j k_0}{2\lambda} \frac{J_0(\lambda \rho)}{\lambda} d\lambda = \frac{-j k_0 \alpha}{2} \int_{\lambda_m}^{\infty} \frac{J_0(\lambda \rho)}{\lambda^2} d\lambda$$

$$= \frac{j k_0 \alpha}{2} \left[ \frac{J_0(\lambda \rho)}{\lambda} \bigg|_{\lambda_m}^{\infty} + \int_{\lambda_m}^{\infty} \frac{\rho J_1(\lambda \rho)}{\lambda} d\lambda \right]$$

$$= \frac{j k_0 \alpha}{2} \left[ -\frac{J_0(\lambda m)}{\lambda m} + \rho \int_{\lambda_m}^{\infty} \left[ \frac{\rho J_0(\lambda \rho) - \frac{d}{d\lambda} J_1(\lambda \rho)}{\lambda} \right] d\lambda \right]$$

$$= \frac{j k_0 \alpha}{2} \left[ -\frac{J_0(\lambda m)}{\lambda m} + \rho J_1(\lambda m) + \rho - \int_{0}^{\lambda_m} \rho^2 J_0(\lambda \rho) d\lambda \right], \quad (A.18)$$

$W^{HE}$ can be rewritten as

$$W^{HE} = \left\{ -\frac{j}{2k_0 \rho} \frac{\epsilon_0}{\epsilon_{av}} + \frac{j k_0 \rho \alpha}{2} \left[ 1 - \frac{J_0(\lambda m \rho)}{\lambda m \rho} - J'_0(\lambda m \rho) \right] \right.$$  

$$+ \int_{0}^{\lambda_m} \left[ \frac{C'_{\Delta e,HE}}{m_{\Delta e,HE} - \frac{k_0^2}{k_m^2} m_{\Delta m,HE}} \frac{1}{\lambda} + \frac{j}{2k_0} \left( \frac{\epsilon_0}{\epsilon_{av}} - k_0^2 \rho^2 \alpha \right) \right] J_0(\lambda \rho) d\lambda \right\} \frac{\mu_0}{\mu_m}. \quad (A.19)$$

A.2 Calculation of $V^{VE1}_{1a}$, $V^{VE1}_{1b}$, $V^{VE1}_{qa}$, $V^{VE1}_{qb}$ and $W^{VE1}$

Following the same approach used in the previous section, $U^{VE1}$ for large $\lambda$ can be approximated as

$$U^{VE1} = \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} ; & n = 2, \\
\frac{1}{\prod_{i=2}^{n-1} U^m_i} \approx \frac{1}{\prod_{i=2}^{n-1} \frac{e^{\Delta e_{i-1}}}{2}} \begin{bmatrix} \frac{\epsilon_i k_0}{j \lambda^2} & \frac{\epsilon_i k_0}{j \lambda^2} \\ \epsilon_i & \epsilon_i \end{bmatrix} = U^\infty_{11} \begin{bmatrix} \frac{\epsilon_{n-1} k_0}{j \lambda^2} & \frac{\epsilon_{n-1} k_0}{j \lambda^2} \\ \epsilon_n & \epsilon_n \end{bmatrix} ; & \text{otherwise}, \end{cases} \quad (A.20)$$
Thus, the integrands for large $\lambda$

\[
U_{11}^\infty = \begin{cases} 
\frac{e^{\lambda z_{n-1}}}{2}; & n = 3 \\
\frac{e^{\lambda (z_{n-2})}}{2^{n-2}} \prod_{i=2}^{n-2} (1 + \frac{\epsilon_i}{\epsilon_{i+1}}); & \text{otherwise.}
\end{cases}
\]  

(A.21)

Therefore, using the above result along with $\xi_i = -j\lambda, \forall i$, for large $\lambda$, one obtains the components in $V_{1a}^{VE}$ and $V_{1b}^{VE}$ functions when $\lambda$ is large as

\[
\frac{U_{21}^{VE} - \xi_0 U_{22}^{VE}}{\Delta m, VE} \simeq -\frac{\epsilon_1}{\epsilon_1 + \epsilon_2} e^{-\lambda z_2},
\]

(A.22)

\[
\frac{U_{11}^{VE} - \xi_0 U_{12}^{VE}}{\Delta m, VE} \simeq \frac{\epsilon_2}{\epsilon_1 + \epsilon_2} e^{-\lambda z_2},
\]

(A.23)

\[
\left\{ \begin{array}{l}
\cos \xi_1(z_2 - |z - z'|) \\
\sin \xi_1(z_2 - |z - z'|) 
\end{array} \right\} \simeq \frac{1}{2} \left[ e^{\lambda(z_2 - |z - z'|)} \pm e^{-\lambda(z_2 - |z - z'|)} \right],
\]

(A.24)

\[
\left\{ \begin{array}{l}
\cos \xi_1(z_2 - (z + z')) \\
\sin \xi_1(z_2 - (z + z')) 
\end{array} \right\} \simeq \frac{1}{2} \left[ e^{\lambda(z_2 - (z + z'))} \pm e^{-\lambda(z_2 - (z + z'))} \right].
\]

(A.25)

Thus, the integrands for large $\lambda$ of $V_{1a}^{VE}$ and $V_{1b}^{VE}$ can be given by

\[
f_{V_{1a}^{VE}} = \frac{j}{2k_0\lambda} \left[ e^{-\lambda |z - z'|} + \frac{\epsilon_2 - \epsilon_1}{\epsilon_2 + \epsilon_1} e^{-\lambda(2z_2 - |z - z'|)} \right],
\]

(A.26)

\[
f_{V_{1b}^{VE}} = \frac{j}{2k_0\lambda} \left[ e^{-\lambda(z + z')} + \frac{\epsilon_2 - \epsilon_1}{\epsilon_2 + \epsilon_1} e^{-\lambda(2z_2 - (z + z'))} \right].
\]

(A.27)

Now using the following identity

\[
\int_0^\infty J_0(\beta x)e^{-\alpha x}dx = \frac{1}{\sqrt{\alpha^2 + \beta^2}} \quad \text{Re} \alpha > |\text{Im} \beta|,
\]

(A.28)

one obtains

\[
\int_{\lambda_m}^{\lambda_0} f_{V_{1a}^{VE}} J_0(\lambda \rho) \lambda d\lambda = \int_0^{\lambda_0} f_{V_{1a}^{VE}} J_0(\lambda \rho) \lambda d\lambda - \int_0^{\lambda_m} f_{V_{1a}^{VE}} J_0(\lambda \rho) \lambda d\lambda
\]

\[
\quad \quad \quad \quad \quad \quad = \frac{j}{2k_0} \left[ \frac{1}{\sqrt{\rho^2 + (z - z')^2}} + \frac{\epsilon_2 - \epsilon_1}{\epsilon_2 + \epsilon_1} \frac{1}{\sqrt{\rho^2 + (2z_2 - |z - z'|)^2}} \right]
\]

\[
\quad \quad \quad \quad \quad \quad - \frac{j}{2k_0} \int_0^{\lambda_m} \left[ e^{-\lambda|z - z'|} + \frac{\epsilon_2 - \epsilon_1}{\epsilon_2 + \epsilon_1} e^{-\lambda(2z_2 - |z - z'|)} \right] J_0(\lambda \rho) d\lambda,
\]

(A.29)
\[
\int_{\lambda_m}^{\infty} f_{V'E_1}^{V,V'} f_{V'}^{V,V'} J_0(\lambda \rho) \lambda d\lambda = \frac{j}{2k_0} \left[ \frac{1}{\sqrt{\rho^2 + (z + z')^2}} + \frac{\epsilon_2 - \epsilon_1}{\epsilon_2 + \epsilon_1} \frac{1}{\sqrt{\rho^2 + [2z_2 - (z + z')]^2}} \right] \\
- \frac{j}{2k_0} \int_{0}^{\lambda_m} \left[ e^{-\lambda(z+z')} + \frac{\epsilon_2 - \epsilon_1}{\epsilon_2 + \epsilon_1} e^{-\lambda(2z_2 - (z + z'))} \right] J_0(\lambda \rho) d\lambda.
\]

(A.30)

To obtain \( f_{V'=V+1}^{V,E_1} \), first noticing that

\[
\mathcal{V}_{q,VE_1}^{V_0} = \prod_{k=q}^{n-1} \mathcal{U}_k^m \\
= \mathcal{U}_q^m \prod_{k=q+1}^{n-1} \mathcal{U}_k^m \\
= \begin{bmatrix}
\cos(\xi_q \Delta z_q) & -j \frac{\epsilon_0 k_0}{\epsilon_q} \sin(\xi_q \Delta z_q) \\
-j \frac{\epsilon_0}{k_0} \sin(\xi_q \Delta z_q) & \frac{\epsilon_0}{\epsilon_q + 1} \cos(\xi_q \Delta z_q)
\end{bmatrix} \mathcal{W} \\
= \begin{bmatrix}
\mathcal{W}_{11} \cos \xi_q \Delta z_q - j \mathcal{W}_{21} \frac{\epsilon_0 k_0}{\epsilon_q} \sin \xi_q \Delta z_q & \mathcal{W}_{12} \cos \xi_q \Delta z_q - j \mathcal{W}_{22} \frac{\epsilon_0 k_0}{\epsilon_q} \sin \xi_q \Delta z_q \\
\mathcal{W}_{21} \frac{\epsilon_0}{\epsilon_q + 1} \cos \xi_q \Delta z_q - j \mathcal{W}_{11} \frac{\epsilon_0}{k_0} \sin \xi_q \Delta z_q & \mathcal{W}_{22} \frac{\epsilon_0}{\epsilon_q + 1} \cos \xi_q \Delta z_q - j \mathcal{W}_{12} \frac{\epsilon_0}{k_0} \sin \xi_q \Delta z_q
\end{bmatrix}.
\]

(A.31)

Since

\[
C_{q}^{m',VE_1} = (\mathcal{V}_{q_1,VE_1}^{V_0} - \frac{\xi_0}{k_0} \mathcal{V}_{q_2,VE_1}^{V_0}) C_{n}^{m',VE_1},
\]

\[
B_{q}^{m',VE_1} = (\mathcal{V}_{q_2,VE_1}^{V_0} - \frac{\xi_0}{k_0} \mathcal{V}_{q_1,VE_1}^{V_0}) C_{n}^{m',VE_1},
\]

and

\[
C_{n}^{m',VE_1} = \frac{\epsilon_2}{\epsilon_1 k_0} j^2 \cos \xi_1 z',
\]

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\(C^m',VE_1\) and \(B^m',VE_1\) can be given by

\[
C^m',VE_1 = \left( v^1_{q,VE_1} - \frac{\xi_n}{k_0} v^2_{q,VE_1} \right) C^m',VE_1
\]

\[
= \left\{ \left( \mathcal{W}_{11} - \frac{\xi_n}{k_0} \mathcal{W}_{12} \right) \cos \xi_q \Delta z_q - \left( \mathcal{W}_{21} - \frac{\xi_n}{k_0} \mathcal{W}_{22} \right) j \frac{\epsilon_q}{\epsilon_{q+1}} \frac{k_0}{\xi_q} \sin \xi_q \Delta z_q \right\} \frac{\epsilon_2}{\epsilon_1 k_0}
\]

\[ j2 \cos \xi_1 z' \]

(A.32)

\[
B^m',VE_1 = \left( v^1_{q,VE_1} - \frac{\xi_n}{k_0} v^2_{q,VE_1} \right) C^m',VE_1
\]

\[
= \left\{ \left( \mathcal{W}_{21} - \frac{\xi_n}{k_0} \mathcal{W}_{22} \right) \frac{\epsilon_q}{\epsilon_{q+1}} \cos \xi_q \Delta z_q - \left( \mathcal{W}_{11} - \frac{\xi_n}{k_0} \mathcal{W}_{12} \right) j \frac{\epsilon_q}{\epsilon_{q+1}} \frac{k_0}{\xi_q} \sin \xi_q \Delta z_q \right\} \frac{\epsilon_2}{\epsilon_1 k_0}
\]

Thus,

\[
C^m',VE_1 \cos \xi_q(z - z_q) + j \frac{k_0}{\xi_q} B^m',VE_1 \sin \xi_q(z - z_q)
\]

\[
= C^m',VE_1 \left\{ \left( \mathcal{W}_{11} - \frac{\xi_n}{k_0} \mathcal{W}_{12} \right) \cos \xi_q \Delta z_q \cos \xi_q(z - z_q) + \sin \xi_q \Delta z_q \sin \xi_q(z - z_q) \right] \]

\[- j \frac{\epsilon_q}{\epsilon_{q+1}} \frac{k_0}{\xi_q} \left( \mathcal{W}_{21} - \frac{\xi_n}{k_0} \mathcal{W}_{22} \right) \sin \xi_q \Delta z_q \cos \xi_q(z - z_q) - \cos \xi_q \Delta z_q \sin \xi_q(z - z_q) \right] \}

\[
= \left\{ \left[ \mathcal{W}_{11} - \frac{\xi_n}{k_0} \mathcal{W}_{12} - \frac{\epsilon_q}{\epsilon_{q+1}} \frac{k_0}{\xi_q} \left( \mathcal{W}_{21} - \frac{\xi_n}{k_0} \mathcal{W}_{22} \right) \right] \frac{e^{i \xi_q(z_q+1-z)}}{2} \right\} C^m',VE_1
\]

\[
+ \left[ \mathcal{W}_{11} - \frac{\xi_n}{k_0} \mathcal{W}_{12} + \frac{\epsilon_q}{\epsilon_{q+1}} \frac{k_0}{\xi_q} \left( \mathcal{W}_{21} - \frac{\xi_n}{k_0} \mathcal{W}_{22} \right) \right] \frac{e^{-i \xi_q(z_q+1-z)}}{2} \right\} C^m',VE_1
\]

\[
= \left\{ \left[ \mathcal{W}_{11} - \frac{\epsilon_q}{\epsilon_{q+1}} \frac{k_0}{\xi_q} \mathcal{W}_{12} \right] \frac{e^{i \xi_q(z_q+1-z)}}{2} + \left( \mathcal{W}_{2} + \frac{\epsilon_q}{\epsilon_{q+1}} \frac{k_0}{\xi_q} \mathcal{W}_{1} \right) \frac{e^{-i \xi_q(z_q+1-z)}}{2} \right\} C^m',VE_1.
\]

(A.34)

For large \(\lambda\),

\[
\mathcal{W} = \begin{cases} 
[1, 0] ; & q + 1 = n, \\
[0, 1] ; & q + 1 \neq n, \\
\mathcal{W}_{11} \frac{1}{j \lambda} \frac{\xi_n - k_0}{\xi_n + k_0} \end{cases} \quad \text{; otherwise,}
\]

(A.35)
where
\[
\mathcal{W}_{11}^\infty = \begin{cases} 
\frac{e^{\lambda \Delta_{s_n-1}}}{2} & n = q + 2 \\
\frac{e^{\lambda (s_n-s_{q+1})}}{2^{n-2}} \prod_{k=q+1}^{n-2} (1 + \frac{\epsilon_k}{\epsilon_{k+1}}) & \text{otherwise},
\end{cases}
\tag{A.36}
\]
thus, one obtains the following approximations
\[
\begin{align*}
\mathcal{W}_1' &= \mathcal{W}_{11} - \frac{\xi_0}{k_0} \mathcal{W}_{12} \simeq \mathcal{W}_{11}^\infty \left( 1 + \frac{\epsilon_{n-1}}{\epsilon_n} \right), \\
\mathcal{W}_2' &= \mathcal{W}_{21} - \frac{\xi_0}{k_0} \mathcal{W}_{22} \simeq \mathcal{W}_{11}^\infty \frac{j \lambda}{k_0} \left( 1 + \frac{\epsilon_{n-1}}{\epsilon_n} \right).
\end{align*}
\tag{A.37}
\]
Using the above results along with (A.20) and \( C_{m',VE}^{m',VE} \simeq \frac{\epsilon_2}{\epsilon_{1k_0}} j (e^{\lambda z'} + e^{-\lambda z'}) \) yields
\[
\begin{align*}
\frac{1}{\Delta_{m,VE}} & \left[ C_{m,VE}^{m',VE} \cos \xi_0 (z - z_q) + \frac{j k_0}{\xi_0} B_{q}^{m',VE} \sin \xi_0 (z - z_q) \right] \\
& \simeq -\frac{\gamma_q}{\lambda} \left[ e^{-\lambda (z-z')} + e^{-\lambda (z+z')} + \alpha_q \left( e^{-\lambda (2z_q+1-z-z')} + e^{-\lambda (2z_q+1-z+z')} \right) \right],
\end{align*}
\tag{A.38}
\]
where \( \gamma_q = 2^{q-1} \prod_{k=1}^{q-1} (1 + \frac{\epsilon_k}{\epsilon_{k+1}})^{-1} \), and \( \alpha_q = \frac{\epsilon_{q+1} - \epsilon_q}{\epsilon_{q+1} + \epsilon_q} \). Therefore, \( f_{V_{qa}}^\infty \) can be given by
\[
f_{V_{qa}}^\infty = \frac{j \gamma_q}{2k_0 \lambda} \left[ e^{-\lambda (z-z')} + e^{-\lambda (z+z')} + \alpha_q \left( e^{-\lambda (2z_q+1-z-z')} + e^{-\lambda (2z_q+1-z+z')} \right) \right],
\tag{A.39}
\]
and it follows that
\[
\begin{align*}
\int_{\lambda_0}^{\lambda_m} f_{V_{qa}}^\infty J_0(\lambda \rho) \lambda d\lambda \\
&= \int_0^{\lambda_m} f_{V_{qa}}^\infty J_0(\lambda \rho) \lambda d\lambda - \int_0^{\lambda_0} f_{V_{qa}}^\infty J_0(\lambda \rho) \lambda d\lambda \\
&= \frac{j \gamma_q}{2k_0} \left[ \frac{1}{\sqrt{\rho^2 + (z-z')^2}} + \frac{1}{\sqrt{\rho^2 + (z+z')^2}} \right] \\
&+ \alpha_q \left( \frac{1}{\sqrt{\rho^2 + (2z_q+1-z-z')^2}} + \frac{1}{\sqrt{\rho^2 + (2z_q+1-z+z')^2}} \right) \\
&- \frac{j \gamma_q}{2k_0} \left[ e^{-\lambda (z-z')} + e^{-\lambda (z+z')} \right] \\
&+ \alpha_q \left( e^{-\lambda (2z_q+1-z-z')} + e^{-\lambda (2z_q+1-z+z')} \right) J_0(\lambda \rho) d\lambda,
\end{align*}
\tag{A.40}
\]
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Likewise, to obtain $f_{Vq}^{\infty \infty V E1}$, first noticing that

$$
(V_{11}^q V_{E1}^q - \frac{\xi_n}{k_0} V_{12}^q V_{E1}^q) \frac{\xi_q}{k_0} j \sin \xi_q (z - z_q) + (V_{21}^q V_{E1}^q - \frac{\xi_n}{k_0} V_{22}^q V_{E1}^q) \cos \xi_q (z - z_q)
$$

\begin{align*}
= \left\{ \mathcal{W}'_1 \cos \xi_q \Delta z_q - \mathcal{W}'_2 j \frac{\xi_q}{k_0} \sin \xi_q \Delta z_q \right\} \frac{\xi_q}{k_0} j \sin \xi_q (z - z_q) \\
+ \left\{ \mathcal{W}'_2 \frac{\xi_q}{k_0} \cos \xi_q \Delta z_q - \mathcal{W}'_1 j \frac{\xi_q}{k_0} \sin \xi_q \Delta z_q \right\} \cos \xi_q (z - z_q) \\
= \mathcal{W}'_2 \frac{\xi_q}{k_0} \cos \xi_q (z_{q+1} - z) - \mathcal{W}'_1 j \frac{\xi_q}{k_0} \sin \xi_q (z_{q+1} - z) \\
= \left( \frac{\xi_q}{k_0} \mathcal{W}'_2 - \frac{\xi_q}{k_0} \mathcal{W}'_1 \right) \frac{e^{j \xi_q (z_{q+1} - z)}}{2} + \left( \frac{\xi_q}{k_0} \mathcal{W}'_2 + \frac{\xi_q}{k_0} \mathcal{W}'_1 \right) \frac{e^{-j \xi_q (z_{q+1} - z)}}{2}.
\end{align*}

(A.41)

Using the above results along with (A.20), (A.37), and

$$
\frac{\sin \xi_1 z'}{\xi_1} \sim \frac{e^{\lambda z'} - e^{-\lambda z'}}{2\lambda}
$$

(A.42)

yields

$$
f_{Vq}^{\infty \infty V E1} = \frac{j \gamma_q}{2k_0 \lambda} \left[ e^{-\lambda (z - z')} - e^{-\lambda (z + z')} - \alpha_q \left( e^{-\lambda (2z_{q+1} - z - z')} - e^{-\lambda (2z_{q+1} - z + z')} \right) \right],
$$

(A.43)

and it follows that

$$
\int_{\lambda_m}^{\infty} f_{Vq}^{\infty \infty V E1} J_0 (\lambda \rho) \lambda d\lambda \\
= \int_0^{\lambda_m} f_{Vq}^{\infty \infty V E1} J_0 (\lambda \rho) \lambda d\lambda - \int_0^{\lambda_m} f_{Vq}^{\infty \infty V E1} J_0 (\lambda \rho) \lambda d\lambda \\
= \frac{j \gamma_q}{2k_0} \left[ \frac{1}{\sqrt{\rho^2 + (z - z')^2}} - \frac{1}{\sqrt{\rho^2 + (z + z')^2}} \right]

- \alpha_q \left( \frac{1}{\sqrt{\rho^2 + (2z_{q+1} - z - z')^2}} - \frac{1}{\sqrt{\rho^2 + (2z_{q+1} - z + z')^2}} \right)

- \frac{j \gamma_q}{2k_0} \int_0^{\lambda_m} \left[ e^{-\lambda (z - z')} - e^{-\lambda (z + z')} \right. \\

- \alpha_q \left( e^{-\lambda (2z_{q+1} - z - z')} - e^{-\lambda (2z_{q+1} - z + z')} \right) \right] J_0 (\lambda \rho) d\lambda,
$$

(A.44)
Likewise, $\mathbf{V}^{m,VE1}$ in (2.77) for large $\lambda$ can be given by

$$\mathbf{V}^{m,VE1} = \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; & m = n, \\ \prod_{i=m}^{n-1} \mathbf{u}_i & \text{otherwise}, \end{cases}$$

where

$$\mathbf{V}_1^\infty = \begin{cases} \frac{e^{\lambda \Delta z_{n-1}}}{2}; & m = n - 1, \\ \frac{e^{\lambda(z_m - z_{m-1})}}{2^{n-m-1}} \prod_{i=m}^{n-2} \left( 1 + \frac{\epsilon_i}{\epsilon_{i+1}} \right); & \text{otherwise}. \end{cases}$$

Thus,

$$\frac{\epsilon_2}{\epsilon_1} \frac{1}{\Delta m,VE1} (\mathbf{V}^{m,VE1}_{21} - \frac{\xi_n}{k_0} \mathbf{V}^{m,VE1}_{22}) \bigg|_{\lambda \rightarrow \infty} \approx -e^{-\lambda z_m} \prod_{i=1}^{m-1} \left( 1 + \frac{\epsilon_i}{\epsilon_{i+1}} \right) = -\alpha_m \frac{e^{-\lambda z_m}}{2^{m+1}},$$

where $\alpha_m = \prod_{i=1}^{m-1} \left( 1 + \frac{\epsilon_i}{\epsilon_{i+1}} \right)^{-1}$. Using the above result along with (A.42) one obtains

$$f_{W,VE1}^\infty = -\alpha_m \frac{2^{m-2}}{jk_0 \lambda} (e^{-\lambda(z_m - z')} - e^{-\lambda(z_m + z')}).$$

Using (A.28) yields

$$\int_{\lambda_m}^{\infty} f_{W,VE1}^\infty J_0(\lambda \rho) \lambda d\lambda = \int_0^{\lambda_m} f_{W,VE1}^\infty J_0(\lambda \rho) \lambda d\lambda - \int_0^{\lambda_m} f_{W,VE1}^\infty J_0(\lambda \rho) \lambda d\lambda
$$

$$= \frac{j \alpha_m 2^{m-2}}{k_0} \left[ \frac{1}{\sqrt{\rho^2 + (z_m - z')^2}} - \frac{1}{\sqrt{\rho^2 + (z_m + z')^2}} \right]
$$

$$- \int_0^{\lambda_m} (e^{-\lambda(z_m - z')} - e^{\lambda(z_m + z')}) J_0(\lambda \rho) d\lambda.$$

### A.3 Calculation of $V_{pa}^{V E p}$, $V_{pb}^{V E p}$, $V_{qa}^{V E p}$, $V_{qb}^{V E p}$ and $W^{V E p}$

Since the multilayered media dyadic Green’s function for this case is quite similar to that for the electric field due to a vertical electric source in the bottommost layer, the approach used to evaluate these functions is almost the same as that in the previous section,
except for some modifications which are needed due to the differences in some parameters and also the introduction of some additional parameters. To obtain $f_{V_{Ep}}^\infty$ and $f_{V_{Ep}}^\infty$, first the matrices $W^{V_{Ep}}$ and $D^{V_{Ep}}$ for large $\lambda$ can be approximated as

$$W^{V_{Ep}} = \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; & n = p + 1, \\ \prod_{i=p+1}^{n-1} U_i^n \simeq \prod_{i=p+1}^{n-1} \frac{e^{\lambda z_{n-1}}}{2} \begin{bmatrix} 1 & \frac{\epsilon_{i} k_0}{\epsilon_{i+1}} \\ \frac{1}{\epsilon_{i}} & \epsilon_{i+1} \end{bmatrix} = W_{11}^\infty \begin{bmatrix} 1 & \frac{\epsilon_{n-1} k_0}{\epsilon_{n}} \\ \frac{1}{\epsilon_{n-1} k_0} & \epsilon_{n} \end{bmatrix} ; & \text{otherwise}, \end{cases}$$

(A.50)

where

$$W_{11}^\infty = \begin{cases} \frac{e^{\lambda z_{i-1}}}{2}; & n = p + 2 \\ \frac{e^{\lambda (zn-zp+1)}}{2^{n-2}} \prod_{i=2}^{n-2} \left(1 + \frac{\epsilon_{i}}{\epsilon_{i+1}} \right); & \text{otherwise}, \end{cases}$$

(A.51)

and

$$D^{V_{Ep}} = \prod_{i=p}^{2} D_i^n \simeq \prod_{i=p}^{2} \frac{e^{\lambda z_{n-1}}}{2} \begin{bmatrix} 1 & \frac{\lambda}{1} \\ \frac{\lambda}{\epsilon_{i-1} k_0} & \epsilon_{i+1} \end{bmatrix} = D_{11}^\infty \begin{bmatrix} 1 & \frac{-\lambda}{\epsilon_{p+1}} \\ \frac{1}{\epsilon_{p+1}} & \epsilon_{p+1} \end{bmatrix}$$

(A.52)

where

$$D_{11}^\infty = \begin{cases} \frac{e^{\lambda z_{1}}}{2}; & p = 2 \\ \frac{e^{\lambda z_{p}}}{2^{p-1}} \prod_{i=p-1}^{2} \left(1 + \frac{\epsilon_{i}}{\epsilon_{i+1}} \right); & \text{otherwise}, \end{cases}$$

(A.53)

respectively. Using the above result along with $\xi_i = -j \lambda, \forall i$, for large $\lambda$, the following approximations can be obtained

$$W_1' = W_{11}^{V_{Ep}} - \frac{\xi_n}{k_0} W_{12}^{V_{Ep}} \simeq W_{11}^\infty \left(1 + \frac{\epsilon_{n-1}}{\epsilon_{n}} \right),$$

$$W_2' = W_{21}^{V_{Ep}} - \frac{\xi_n}{k_0} W_{22}^{V_{Ep}} \simeq W_{11}^\infty \lambda \left(1 + \frac{\epsilon_{n-1}}{\epsilon_{n}} \right),$$

(A.54)

$$\Delta^{m,V_{Ep}} \simeq -\frac{j \lambda}{k_0} D_{11}^\infty W_{11}^\infty \left(1 + \frac{\epsilon_{n-1}}{\epsilon_{n}} \right) \left(1 + \frac{\epsilon_{p}}{\epsilon_{p+1}} \right) \left(1 + \frac{\epsilon_{p}}{\epsilon_{p-1}} \right) \frac{e^{\lambda z_{p}}}{2}.$$  

(A.55)

It follows that

$$\frac{\xi_p}{k_0} D_{11}^{V_{Ep}} W_{11}' - \frac{\epsilon_{p} k_0}{\epsilon_{p+1}} D_{21}^{V_{Ep}} W_{21}' \simeq \frac{j \lambda}{k_0} D_{11}^\infty W_{11}^\infty \left(1 + \frac{\epsilon_{n-1}}{\epsilon_{n}} \right) \left(1 + \frac{\epsilon_{p}}{\epsilon_{p+1}} \right) \left(1 + \frac{\epsilon_{p}}{\epsilon_{p-1}} \right).$$

(A.56)
Using the above results along with
\[
\frac{\varepsilon_p}{\varepsilon_{p+1}} D_{11}^{V_{Ep}} W'_2 - D_{21}^{V_{Ep}} W'_1 \approx \frac{j \lambda}{k_0} D_{11}^{\infty} W_1^{\infty} \left( 1 + \frac{\varepsilon_{n-1}}{\varepsilon_n} \right) \left( \frac{\varepsilon_p}{\varepsilon_{p-1}} + \frac{\varepsilon_p}{\varepsilon_{p+1}} \right). \tag{A.57}
\]

It is noted that the last term in \( f \) and \( \epsilon \) can be found to be
\[
f_{V_{pa}}^{\infty} \text{ and } f_{V_{pb}}^{\infty} \text{ can be found to be}
\]
\[
f_{V_{pa}}^{\infty} = \frac{j}{2k_0 \lambda} \left[ e^{-\lambda(z-z')} + \frac{\epsilon_{p-1} - \epsilon_p}{\epsilon_{p-1} + \epsilon_p} e^{-\lambda(2\Delta z_p - |z-z'|)} \right], \tag{A.60}
\]
and
\[
f_{V_{pb}}^{\infty} = \frac{j}{2k_0 \lambda} \left[ \frac{\epsilon_{p-1} - \epsilon_p}{\epsilon_{p-1} + \epsilon_p} e^{-\lambda(z+z'-2\rho_p)} + \frac{\epsilon_{p+1} - \epsilon_p}{\epsilon_{p+1} + \epsilon_p} e^{-\lambda(2\Delta z_p - (z'-z)}) + e^{-\lambda(z+z'-2\rho_{p-1})} \right]. \tag{A.61}
\]

It is noted that the last term in \( f_{V_{Ep}}^{\infty} \) is included to account for the case when \( \epsilon_p \) is almost the same as \( \epsilon_{p-1} \). Let \( \kappa_1 = \frac{\epsilon_{p-1}-\epsilon_p}{\epsilon_{p-1}+\epsilon_p} \) and \( \kappa_2 = \frac{\epsilon_{p+1}-\epsilon_p}{\epsilon_{p+1}+\epsilon_p} \), then one obtains
\[
\int_{\lambda_m}^{\infty} f_{V_{pa}}^{\infty} J_0(\lambda \rho) \lambda d\lambda = \int_{0}^{\infty} f_{V_{pa}}^{\infty} J_0(\lambda \rho) \lambda d\lambda - \int_{0}^{\lambda_m} f_{V_{pa}}^{\infty} J_0(\lambda \rho) \lambda d\lambda
\]
\[
= \frac{j}{2k_0} \left[ \frac{1}{\sqrt{\rho^2 + (z-z')^2}} + \kappa_1 \kappa_2 \frac{1}{\sqrt{\rho^2 + (2\Delta z_p - |z-z'|)^2}} \right]
- \frac{j}{2k_0} \int_{0}^{\lambda_m} \left[ e^{-\lambda|z-z'|} + \kappa_1 \kappa_2 e^{-\lambda(2\Delta z_p - |z-z'|)} \right] J_0(\lambda \rho) d\lambda, \tag{A.62}
\]
\[
\int_{\lambda_m}^{\infty} f_{V_{pb}}^\infty J_0(\lambda \rho) \lambda d\lambda \\
= \int_0^{\infty} f_{V_{pb}}^\infty J_0(\lambda \rho) \lambda d\lambda - \int_0^{\lambda_m} f_{V_{pb}}^\infty J_0(\lambda \rho) \lambda d\lambda \\
= \frac{j}{2k_0} \left[ \frac{\kappa_1}{\sqrt{\rho^2 + (z + z' - 2z_p)^2}} + \frac{\kappa_2}{\sqrt{\rho^2 + (2z_{p+1} - z - z')^2}} + \frac{1}{\sqrt{\rho^2 + (z + z' - 2z_{p-1})^2}} \right] \\
- \frac{j}{2k_0} \int_{\lambda_m}^{\infty} \left[ \kappa_1 e^{-\lambda(z+z'-2z_p)} + \kappa_2 e^{-\lambda(2z_{p+1} - z - z')} + e^{-\lambda(z+z'-2z_{p-1})} \right] J_0(\lambda \rho) d\lambda. \tag{A.63}
\]

To obtain \( f_{V_{qa}}^\infty \), first noticing that

\[
\begin{align*}
V_{qa}^{V_{Ep}} &= n - 1 \prod_{k=q}^{n-1} U_k^m \\
&= u_q^m \prod_{k=q+1}^{n-1} U_k^m \\
&= \begin{bmatrix}
\cos(\xi_q \Delta z_q) & -j \frac{\epsilon_q k_0}{\epsilon_{q+1}} \sin(\xi_q \Delta z_q) \\
-j \frac{\epsilon_q k_0}{\epsilon_{q+1}} \sin(\xi_q \Delta z_q) & \frac{\epsilon_q}{\epsilon_{q+1}} \cos(\xi_q \Delta z_q)
\end{bmatrix} \begin{bmatrix}
\chi_{11} \cos \xi_q \Delta z_q - j \chi_{21} \frac{\epsilon_q}{\epsilon_{q+1}} \cos \xi_q \Delta z_q \\
\chi_{21} \frac{\epsilon_q}{\epsilon_{q+1}} \cos \xi_q \Delta z_q - j \chi_{11} \frac{\epsilon_q k_0}{\epsilon_{q+1}} \sin \xi_q \Delta z_q
\end{bmatrix} \\
&= \begin{bmatrix}
\chi_{11} \cos \xi_q \Delta z_q - j \chi_{21} \frac{\epsilon_q}{\epsilon_{q+1}} \cos \xi_q \Delta z_q \\
\chi_{21} \frac{\epsilon_q}{\epsilon_{q+1}} \cos \xi_q \Delta z_q - j \chi_{11} \frac{\epsilon_q k_0}{\epsilon_{q+1}} \sin \xi_q \Delta z_q
\end{bmatrix} \\
&= \begin{bmatrix}
\chi_{11} \cos \xi_q \Delta z_q - j \chi_{21} \frac{\epsilon_q}{\epsilon_{q+1}} \cos \xi_q \Delta z_q \\
\chi_{21} \frac{\epsilon_q}{\epsilon_{q+1}} \cos \xi_q \Delta z_q - j \chi_{11} \frac{\epsilon_q k_0}{\epsilon_{q+1}} \sin \xi_q \Delta z_q
\end{bmatrix} \\
&= \begin{bmatrix}
\chi_{11} \cos \xi_q \Delta z_q - j \chi_{21} \frac{\epsilon_q}{\epsilon_{q+1}} \cos \xi_q \Delta z_q \\
\chi_{21} \frac{\epsilon_q}{\epsilon_{q+1}} \cos \xi_q \Delta z_q - j \chi_{11} \frac{\epsilon_q k_0}{\epsilon_{q+1}} \sin \xi_q \Delta z_q
\end{bmatrix}. \tag{A.64}
\end{align*}
\]

Since

\[ C_{q}^{m',V_{Ep}} = (V_{l1}^{q,V_{Ep}} - \frac{\xi_q}{k_0} V_{l2}^{q,V_{Ep}}) C_{n}^{m',V_{Ep}}, \]

and

\[ B_{q}^{m',V_{Ep}} = (V_{l1}^{q,V_{Ep}} - \frac{\xi_q}{k_0} V_{l2}^{q,V_{Ep}}) C_{n}^{m',V_{Ep}}, \]

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$C_{q',VEp}^m$ and $B_{q',VEp}^m$ can be given by

\[
C_{q',VEp}^m = \left( \nu_{q,VEp}^{q,VEp} - \frac{\xi_n}{k_0} \nu_{q,VEp}^{q,VEp} \right) C_n^{m',VEp}
\]

\[
= \left\{ \left( \chi_{11} - \frac{\xi_n}{k_0} \chi_{12} \right) \cos \xi_q \Delta z_q - \left( \chi_{21} - \frac{\xi_n}{k_0} \chi_{22} \right) j \frac{\epsilon_q}{\epsilon_{q+1} \xi_q} \sin \xi_q \Delta z_q \right\} C_n^{m',VEp}
\]

(A.65)

\[
B_{q',VEp}^m = \left( \nu_{q,VEp}^{q,VEp} - \frac{\xi_n}{k_0} \nu_{q,VEp}^{q,VEp} \right) C_n^{m',VEp}
\]

\[
= \left\{ \left( \chi_{21} - \frac{\xi_n}{k_0} \chi_{22} \right) \cos \xi_q \Delta z_q - \left( \chi_{11} - \frac{\xi_n}{k_0} \chi_{12} \right) j \frac{\epsilon_q}{\epsilon_{q+1} \xi_q} \sin \xi_q \Delta z_q \right\} C_n^{m',VEp}
\]

(A.66)

Thus,

\[
C_{q',VEp}^m \cos \xi_q (z - z_q) + j \frac{k_0}{\epsilon_q} B_{q',VEp}^m \sin \xi_q (z - z_q)
\]

\[
= C_n^{m',VEp} \left\{ \left( \chi_{11} - \frac{\xi_n}{k_0} \chi_{12} \right) \cos \xi_q \Delta z_q \cos \xi_q (z - z_q) + \sin \xi_q \Delta z_q \sin \xi_q (z - z_q) \right\}
\]

\[
- j \frac{\epsilon_q}{\epsilon_{q+1} \xi_q} \left( \chi_{21} - \frac{\xi_n}{k_0} \chi_{22} \right) \sin \xi_q \Delta z_q \cos \xi_q (z - z_q) - \cos \xi_q \Delta z_q \sin \xi_q (z - z_q) \right\}
\]

\[
= \left\{ \left[ \chi_{11} - \frac{\xi_n}{k_0} \chi_{12} - \frac{\epsilon_q}{\epsilon_{q+1} \xi_q} \left( \chi_{21} - \frac{\xi_n}{k_0} \chi_{22} \right) \right] \frac{e^{j \xi_q (z_q + 1 - z)}}{2}
\]

\[
+ \left[ \chi_{11} - \frac{\xi_n}{k_0} \chi_{12} + \frac{\epsilon_q}{\epsilon_{q+1} \xi_q} \left( \chi_{21} - \frac{\xi_n}{k_0} \chi_{22} \right) \right] \frac{e^{-j \xi_q (z_q + 1 - z)}}{2} \right\} C_n^{m',VEp}
\]

\[
= \left[ \left( \chi'_{11} - \frac{\epsilon_q}{\epsilon_{q+1} \xi_q} \chi'_{12} \right) \frac{e^{j \xi_q (z_q + 1 - z)}}{2} + \left( \chi'_{21} - \frac{\epsilon_q}{\epsilon_{q+1} \xi_q} \chi'_{12} \right) \frac{e^{-j \xi_q (z_q + 1 - z)}}{2} \right] C_n^{m',VEp}
\]

(A.67)

For large $\lambda$,

\[
\chi = \begin{cases} 
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & ; \quad q + 1 = n, \\
\chi_{11}^{\infty} \begin{bmatrix} 1 & \frac{\epsilon_{n-1} k_0}{\epsilon_n} \\ j_k \frac{\epsilon_{n-1}}{\epsilon_n} \end{bmatrix} & ; \quad \text{otherwise},
\end{cases}
\]

(A.68)
where
\[ X_{11}^\infty = \begin{cases} \frac{e^{\lambda \Delta z_{n-1}}}{2} & n = q + 2 \\ e^{\lambda z_{n-1}} \prod_{k=q+1}^{n} (1 + \frac{\epsilon_k}{\epsilon_k+1}) & \text{otherwise} \end{cases} \] (A.69)
thus, one obtains the following approximations
\[
X_1' = X_{11} - \frac{\xi_0}{k_0} X_{12} \approx X_{11}^\infty \left( 1 + \frac{\epsilon_{n-1}}{\epsilon_n} \right), \\
X_2' = X_{21} - \frac{\xi_0}{k_0} X_{22} \approx X_{11}^\infty \frac{f\lambda}{k_0} \left( 1 + \frac{\epsilon_{n-1}}{\epsilon_n} \right). \tag{A.70}
\]
Using the above results along with (A.50), (A.52) and
\[
C_{m', VEp} \approx -\frac{j}{2k_0} D_{11}^\infty \left[ \left( 1 + \frac{\epsilon_p}{\epsilon_{p-1}} \right) e^{\lambda(z' - z_p)} + \left( 1 - \frac{\epsilon_p}{\epsilon_{p-1}} \right) e^{-\lambda(z' - z_p)} \right]
\]
yields
\[
\frac{1}{\Delta m, VEp} \left[ C_{q, VEp} \cos \xi_q(z - z_q) + \frac{jk_0}{\xi_q} B_{m', VEp} \sin \xi_q(z - z_q) \right] \\
\approx -\frac{\gamma_{pq}}{\lambda} \left[ e^{-\lambda(z - z')} + \alpha_q e^{-\lambda(2z_{q+1} - z - z')} + \alpha_p \left( e^{-\lambda(z + z' - 2z_q)} + \alpha_q e^{-\lambda(2z_{q+1} - z + z' - 2z_q)} \right) \right], \tag{A.71}
\]
where \( \gamma_{pq} = 2^{q-p-1} \prod_{k=p}^{q-1} (1 + \frac{\epsilon_k}{\epsilon_{k+1}})^{-1} \), \( \alpha_p = \frac{\epsilon_{p-1} - \epsilon_p}{\epsilon_{p-1} + \epsilon_p} \), and \( \alpha_q = \frac{\epsilon_{q+1} - \epsilon_q}{\epsilon_{q+1} + \epsilon_q} \). Therefore, \( f_{V_{pq}}^\infty \) can be given by
\[
f_{V_{pq}}^\infty = -\frac{j \gamma_q}{2k_0 \lambda} \left[ e^{-\lambda(z - z')} + \alpha_q e^{-\lambda(2z_{q+1} - z - z')} \\
+ \alpha_p \left( e^{-\lambda(z + z' - 2z_q)} + \alpha_q e^{-\lambda(2z_{q+1} - z + z' - 2z_q)} \right) \right], \tag{A.72}
\]
and it follows that

\[
\int_{\lambda_m}^{\infty} f_{V_{qb}}^\infty J_0(\lambda \rho) \lambda d\lambda = \int_0^{\lambda_m} f_{V_{qb}}^\infty J_0(\lambda \rho) \lambda d\lambda - \int_0^{\lambda_m} f_{V_{qa}}^\infty J_0(\lambda \rho) \lambda d\lambda
\]

\[
\begin{align*}
&= -\frac{j \gamma_{pq}}{2k_0} \left[ \frac{1}{\sqrt{\rho^2 + (z - z')^2}} + \frac{\alpha_q}{\sqrt{\rho^2 + (2z_{q+1} - z - z')^2}} \right] \\
&\quad + \frac{j \gamma_{pq}}{2k_0} \int_0^{\lambda_m} \left[ e^{-\lambda(z-z')} + \alpha_q e^{-\lambda(2z_{q+1}-z-z')} \right] J_0(\lambda \rho) d\lambda,
\end{align*}
\]

Similarly, to obtain \( f_{V_{qb}}^\infty \), first noticing that

\[
(V_{11}^{q,V_{Ep}} - \frac{\xi_n}{k_0} V_{12}^{q,V_{Ep}}) \frac{\xi_q}{k_0} j \sin \xi_q(z - z_q) + (V_{21}^{q,V_{Ep}} - \frac{\xi_n}{k_0} V_{22}^{q,V_{Ep}}) \cos \xi_q(z - z_q) = \begin{cases} \mathcal{X}'_1 \cos \xi_q(\Delta z_q) - \mathcal{X}'_2 \frac{\xi_q}{\epsilon_{q+1}} \frac{\xi_k}{\xi_q} j \sin \xi_q(\Delta z_q) + \frac{\xi_k}{k_0} j \sin \xi_q(z - z_q) \\
\mathcal{X}'_2 \frac{\xi_q}{\epsilon_{q+1}} \sin \xi_q(z_{q+1} - z) - \mathcal{X}'_1 \frac{\xi_k}{k_0} \sin \xi_q(z_{q+1} - z) 
\end{cases}
\]

\[
= \left( \frac{\xi_q}{\epsilon_{q+1}} \mathcal{X}'_2 - \frac{\xi_k}{k_0} \mathcal{X}'_1 \right) \frac{e^{\xi_q(\Delta z_q - z)}}{2} + \left( \frac{\xi_q}{\epsilon_{q+1}} \mathcal{X}'_2 + \frac{\xi_k}{k_0} \mathcal{X}'_1 \right) \frac{e^{-\xi_q(z_{q+1} - z)}}{2}.
\]

Using the above results along with (A.50), (A.52), (A.70), and

\[
\left( D_{21}^{V_{Ep}} \frac{\cos \xi_q(z' - z_p)}{\xi_p^2} + \frac{D_{11}^{V_{Ep}}}{k_0} j \sin \xi_p(z' - z_p) \right) \approx \frac{j k_0}{2\lambda} D_{11}^{\infty} \left[ \left( 1 + \frac{\epsilon_p}{\epsilon_{p-1}} \right) e^{\lambda(z'-z_p)} - \left( 1 - \frac{\epsilon_p}{\epsilon_{p-1}} \right) e^{-\lambda(z'-z_p)} \right]
\]

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yields

\[-\frac{1}{\Delta m, V_{Ep}} \left( D_{21}^{V_{Ep}} \cos \xi_p(z' - z_p) + D_{11}^{V_{Ep}} \frac{j}{k_0} \sin \xi_p(z' - z_p) \right) \times \left( V_{11}^{q, V_{Ep}} - \frac{\xi_m}{k_0} y_{q, V_{Ep}} V_{12}^{q, V_{Ep}} \right) \frac{\xi_p}{k_0} j \sin \xi_q(z - z_q) + \left( V_{21}^{q, V_{Ep}} - \frac{\xi_m}{k_0} y_{q, V_{Ep}} V_{22}^{q, V_{Ep}} \right) \cos \xi_q(z - z_q) \right] \\
\simeq -\frac{j \gamma p q}{\lambda} \left[ e^{-\lambda(z-z')} - \alpha_q e^{-\lambda(2z_{q+1} - z' - z')} - \alpha_p \left( e^{-\lambda(z+z'-2z_q)} - \alpha_q e^{-\lambda(2z_{q+1} - z + z' - 2z_q)} \right) \right]. \quad (A.76)

Therefore, \( f_{V_{q b}}^{\infty, V_{Ep}} \) can be given by

\[ f_{V_{q b}}^{\infty, V_{Ep}} = -\frac{j \gamma p q}{2k_0 \lambda} \left[ \frac{1}{\sqrt{\rho^2 + (z - z')^2}} - \frac{\alpha_q}{\sqrt{\rho^2 + (2z_{q+1} - z - z')^2}} \right] \\
- \alpha_p \left( \frac{1}{\sqrt{\rho^2 + (z + z' - 2z_q)^2}} - \frac{\alpha_q}{\sqrt{\rho^2 + (2z_{q+1} - z + z' - 2z_q)^2}} \right) \right] \\
+ \frac{j \gamma p q}{2k_0} \int_{\lambda_m}^{\lambda_m} \left[ e^{-\lambda(z-z')} - \alpha_q e^{-\lambda(2z_{q+1} - z' - z')} \\
- \alpha_p \left( e^{-\lambda(z+z'-2z_q)} - \alpha_q e^{-\lambda(2z_{q+1} - z + z' - 2z_q)} \right) \right] J_0(\lambda \rho) d\lambda, \quad (A.78)

Likewise, \( \mathbf{V}^{m, V_{Ep}} \) in (2.134) for large \( \lambda \) can be given by

\[
\mathbf{V}^{m, V_{Ep}} = \begin{cases} 
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & m = n, \\
\prod_{i=m}^{n-1} u_i^{m} \simeq \prod_{i=m}^{n-1} e^{\lambda \Delta z_{i-1}} - \frac{1}{2} \begin{bmatrix} \frac{\epsilon_{i} k_0}{j \epsilon_{i+1} \lambda} & \frac{\epsilon_{i} k_0}{j \epsilon_{i+1} \lambda} \\ \frac{\epsilon_{i} k_0}{j \epsilon_{i+1} \lambda} & \frac{\epsilon_{i} k_0}{j \epsilon_{i+1} \lambda} \end{bmatrix} \left( V_{11}^{\infty} \left[ \begin{bmatrix} 1 & \frac{\epsilon_{n-1} k_0}{j \epsilon_{n-1} \lambda} \\ \frac{\epsilon_{n-1} k_0}{j \epsilon_{n-1} \lambda} \end{bmatrix} \right] \right) \end{cases}
\]
\quad \text{otherwise,} \quad (A.79)
where

\[ \mathcal{V}_{11}^{\infty} = \begin{cases} e^{\lambda (\Delta m - 1)} & m = n - 1 \\ \frac{e^{\lambda (z_m - z_m)}}{2^{n-m}} & \prod_{i=m}^{n-2} (1 + i_{i+1}) \end{cases}; \text{ otherwise.} \]  

(A.80)

Using the above result along with (A.75) yields

\[ \frac{1}{\Delta_{m,VEp}} \left( \mathcal{V}_{21}^{m,VEp} - \frac{\xi_n}{k_0} \mathcal{V}_{22}^{m,VEp} \right) \left( -D_{21}^{VEp} \cos \frac{\xi_p (z'_p - z_p)}{\xi_p^2} + \frac{D_{11}^{VEp}}{k_0} j \sin \frac{\xi_p (z'_p - z_p)}{\xi_p} \right) \]

\[ \cong \int \frac{\gamma_{pm}}{\lambda} \left( e^{-\lambda (z_m - z')} - \alpha_p e^{-\lambda (z_m + z' - 2z_p)} \right) \]

(A.81)

where \( \gamma_{pm} = 2^{m-p-1} \prod_{k=p}^{m-1} (1 + \frac{\epsilon_k}{\epsilon_{k+1}})^{-1} \), and \( \alpha_p = \frac{\epsilon_{p-1} - \epsilon_p}{\epsilon_{p-1} + \epsilon_p} \). Thus, \( f_{WVEp}^{\infty} \) can be given by

\[ f_{WVEp}^{\infty} = \int \frac{\gamma_{pm}}{\lambda} \left( e^{-\lambda (z_m - z')} - \alpha_p e^{-\lambda (z_m + z' - 2z_p)} \right) \]

(A.82)

Using (A.28) yields

\[ \int_{\lambda_m}^{\infty} f_{WVEp}^{\infty} J_0(\lambda \rho) \lambda d\lambda = \int_{0}^{\infty} f_{WVEp}^{\infty} J_0(\lambda \rho) \lambda d\lambda - \int_{0}^{\lambda_m} f_{WVEp}^{\infty} J_0(\lambda \rho) \lambda d\lambda \]

\[ = \int \frac{\gamma_{pm}}{k_0} \left[ \frac{1}{\sqrt{\rho^2 + \left( z_m - z' \right)^2}} - \frac{\alpha_p}{\sqrt{\rho^2 + \left( z_m + z' - 2z_p \right)^2}} \int_{0}^{\lambda_m} \left( e^{-\lambda (z_m - z')} - \alpha_p e^{\lambda (z_m + z' - 2z_p)} \right) J_0(\lambda \rho) d\lambda \right]. \]  

(A.83)

### A.4 Calculation of \( U^{HM}, W^{HM}, V^{HM}, S^{HM} \) and \( T^{HM} \)

Applying the same approximation used in the previous sections yields \( U^{m,HM} \) and \( U^{c,HM} \) for large \( \lambda \) as follows

\[ U^{m,HM} = \prod_{i=1}^{n-1} U_i^m \simeq \prod_{i=1}^{n-1} \frac{e^{\lambda (\Delta m - 1)}}{2} \left[ \frac{\epsilon_k}{\epsilon_{i+1}} \right] \]

\[ = U_{11}^{m,HM} \left[ \frac{1}{\lambda} \frac{\epsilon_{n-1} k_0}{\epsilon_n} \right], \]  

(A.84)

and

\[ U^{c,HM} = \prod_{i=1}^{n-1} U_i^c \simeq U_{11}^{c,HM} \left[ \frac{1}{\lambda} \frac{\mu_{n-1} k_0}{\mu_n} \right], \]  

(A.85)

where

\[ U_{11}^{\infty,m} = \begin{cases} \frac{e^{\lambda (\Delta m - 1)}}{2} & n = 2 \\ \frac{e^{\lambda (z_m - z_1)}}{2^{n-1}} & \prod_{i=1}^{n-2} (1 + \frac{\epsilon_i}{\epsilon_{i+1}}) \end{cases}; \text{ otherwise,} \]  

(A.86)

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and
\[ U_{11}^{\infty,e} = \begin{cases} \frac{e^{\lambda z_n-1}}{2}; & n = 2 \\ \frac{e^{\lambda (z_n-z_1)}}{2^{n-1}} \prod_{i=1}^{n-2} (1 + \frac{\mu_i}{\mu_{i+1}}); & \text{otherwise.} \end{cases} \]  
(A.87)

Likewise, \( V_{q,H,M}^{mq} \) and \( V_{q,H,M}^{eq} \) for large \( \lambda \) can be given by
\[ V_{q,H,M}^{mq} = \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; & q = n, \\ \prod_{k=q}^{n-1} U_k^m \simeq \prod_{k=q}^{n-1} \frac{e^{\lambda \Delta z_{k-1}}}{2} \frac{1}{\frac{\epsilon_k k_0}{\epsilon_{k+1}}} \frac{\epsilon_{k-1} k_0}{\epsilon_{k+1}} \end{cases} = V_{11}^{\infty, eq} \begin{bmatrix} 1 & \frac{e_{n-1} k_0}{e_n} \frac{\epsilon_{n-1} k_0}{\epsilon_n} \end{bmatrix}; & \text{otherwise,} \]  
(A.88)

and
\[ V_{q,H,M}^{eq} = \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; & q = n, \\ \prod_{k=q}^{n-1} U_k^e \simeq \prod_{k=q}^{n-1} \frac{e^{\lambda \Delta z_{k-1}}}{2} \frac{1}{\frac{\epsilon_k k_0}{\epsilon_{k+1}}} \frac{\epsilon_{k-1} k_0}{\epsilon_{k+1}} \end{cases} = V_{11}^{\infty, eq} \begin{bmatrix} 1 & \frac{e_{n-1} k_0}{e_n} \frac{\epsilon_{n-1} k_0}{\epsilon_n} \end{bmatrix}; & \text{otherwise,} \]  
(A.89)

where
\[ V_{11}^{\infty, mq} = \begin{cases} \frac{e^{\lambda \Delta z_{n-1}}}{2}; & q = n - 1 \\ \frac{e^{\lambda (z_n-z_q)}}{2^{n-q}} \prod_{k=q}^{n-2} (1 + \frac{\epsilon_k}{\epsilon_{k+1}}); & \text{otherwise.} \end{cases} \]  
(A.90)

\[ V_{11}^{\infty, eq} = \begin{cases} \frac{e^{\lambda \Delta z_{n-1}}}{2}; & q = n - 1 \\ \frac{e^{\lambda (z_n-z_q)}}{2^{n-q}} \prod_{k=q}^{n-2} (1 + \frac{\mu_k}{\mu_{k+1}}); & \text{otherwise.} \end{cases} \]  
(A.91)

Using the above results, the large argument approximation for the integrand of \( U_{H,M} \) can then be given as
\[ \frac{C_{m}^{e,H,M}}{\Delta e_{H,M}} \bigg|_{\lambda \to \infty} \approx \frac{e^{-\lambda z_m}}{2^{-m+1}} \prod_{i=1}^{m-1} (1 + \frac{\mu_i}{\mu_{i+1}})^{-1} \]
\[ = \alpha^e \frac{e^{-\lambda z_m}}{2^{-m+1}}, \]
\[ (A.92) \]

where \( \alpha^e = \prod_{i=1}^{m-1} (1 + \frac{\mu_i}{\mu_{i+1}})^{-1} \). Hence, the large argument of \( f(\xi_i, \cdots, \xi_n) \) for \( U_{H,M} \) can be given by
\[ f_{U_{H,M}}^{\infty} = \alpha^e \frac{e^{-\lambda z_m}}{2^{-m+1} k_0^2}, \]
\[ (A.93) \]
Now, using the following identity,

$$\int_0^\infty J_0(\beta x)e^{-\alpha x}dx = \frac{\alpha}{(\alpha^2 + \beta^2)^{3/2}} \quad \text{Re } \alpha > |\text{Im } \beta|,$$  \hspace{1cm} (A.94)

yields

$$\int_{\lambda_m}^\infty f_{UHM}^* J_0(\lambda \rho) \lambda d\lambda = \int_0^{\lambda_m} f_{UHM}^* J_0(\lambda \rho) \lambda d\lambda - \int_0^\infty f_{UHM}^* J_0(\lambda \rho) \lambda d\lambda$$

$$= \frac{\alpha e^{\nu_{m-1}}}{k_0^2} \left[ \frac{z_m}{(z_m^2 + \rho^2)^{3/2}} - \int_0^{\lambda_m} e^{-\lambda z_m} J_0(\lambda \rho) \lambda d\lambda \right].$$  \hspace{1cm} (A.95)

Likewise, \( \frac{B_{m,HM}}{\Delta_{m,HM}} \) for large \( \lambda \) can be approximated as

$$\left. \frac{B_{m,HM}'}{\Delta_{m,HM}} \right|_{\lambda \to \infty} \approx \frac{e^{-\lambda z_m}}{2^{m+1}} \prod_{i=1}^{m-1} \left( 1 + \frac{\epsilon_i}{\epsilon_{i+1}} \right)^{-1}$$

$$= \alpha^m e^{-\lambda z_m} \frac{2^{-m+1}}{2^{m+1}}.$$  \hspace{1cm} (A.96)

where \( \alpha^m = \prod_{i=1}^{m-1} \left( 1 + \frac{\epsilon_i}{\epsilon_{i+1}} \right)^{-1} \). Therefore, the large argument of \( f(\xi, \cdots, \xi_n) \) for \( W_{HM} \) can be given by

$$f_{W_{HM}}^\infty = \left( \alpha^e - \frac{\epsilon_1}{\epsilon_m} \alpha^m \right) \frac{1}{k_0} \frac{1}{2^{-m+1}} e^{-\lambda z_m}.$$  \hspace{1cm} (A.97)

Applying the following identity

$$\int_0^\infty J_1(\beta x)e^{-\alpha x}dx = \frac{\sqrt{\alpha^2 + \beta^2} - \alpha}{\beta \sqrt{\alpha^2 + \beta^2}} \quad \text{Re } (\alpha \pm i\beta) > 0,$$  \hspace{1cm} (A.98)

one obtains

$$\int_{\lambda_m}^\infty f_{W_{HM}}^\infty J_1(\lambda \rho) d\lambda$$

$$= \int_0^{\lambda_m} f_{W_{HM}}^\infty J_1(\lambda \rho) d\lambda - \int_0^\infty f_{W_{HM}}^\infty J_1(\lambda \rho) d\lambda$$

$$= \frac{2^{m-1}}{k_0} \left( \alpha^e - \frac{\epsilon_1}{\epsilon_m} \alpha^m \right) \left[ \frac{\sqrt{z_m^2 + \rho^2} - z_m}{\rho \sqrt{z_m^2 + \rho^2}} - \int_0^{\lambda_m} e^{-\lambda z_m} J_1(\lambda \rho) \lambda d\lambda \right].$$  \hspace{1cm} (A.99)
For $V_q^{HM}$, first notice that

\[
\mathbf{v}^{mq,HM} = \prod_{k=q}^{n-1} \mathbf{u}^m_k = \mathbf{u}^m_q \prod_{k=q+1}^{n-1} \mathbf{u}^m_k
\]

\[
\mathbf{v}^{mq,HM} = \left[ \begin{array}{c}
\cos(\xi_q \Delta z_q) - j \frac{\xi_q k_0}{\epsilon_{q+1} \xi_q} \sin(\xi_q \Delta z_q) \\
- j \frac{\xi_q}{k_0} \sin(\xi_q \Delta z_q) - \frac{\epsilon_q}{\epsilon_{q+1}} \cos(\xi_q \Delta z_q)
\end{array} \right] \mathbf{W}
\]

\[
\mathbf{v}^{mq,HM} = \left[ \begin{array}{c}
\mathbf{W}_{11} \cos \xi_q \Delta z_q - j \mathbf{W}_{21} \frac{\epsilon_q}{\epsilon_{q+1}} \frac{k_0}{\xi_q} \sin \xi_q \Delta z_q \\
\mathbf{W}_{21} \frac{\epsilon_q}{\epsilon_{q+1}} \cos \xi_q \Delta z_q - j \mathbf{W}_{11} \frac{\epsilon_q}{\epsilon_{q+1}} \frac{k_0}{\xi_q} \sin \xi_q \Delta z_q
\end{array} \right].
\]

(A.100)

Then, $C_q^{m',HM}$ and $B_q^{m',HM}$ can be given by

\[
C_q^{m',HM} = \mathbf{v}^{mq,HM} - \frac{\xi_n}{k_0} \mathbf{v}^{mq,HM}
\]

\[
C_q^{m',HM} = \left( \mathbf{W}_{11} - \frac{\xi_n}{k_0} \mathbf{W}_{12} \right) \cos \xi_q \Delta z_q - \left( \mathbf{W}_{21} - \frac{\xi_n}{k_0} \mathbf{W}_{22} \right) j \frac{\epsilon_q}{\epsilon_{q+1}} \frac{k_0}{\xi_q} \sin \xi_q \Delta z_q,
\]

(A.101)

\[
B_q^{m',HM} = \mathbf{v}^{mq,HM} - \frac{\xi_n}{k_0} \mathbf{v}^{mq,HM}
\]

\[
B_q^{m',HM} = \left( \mathbf{W}_{21} - \frac{\xi_n}{k_0} \mathbf{W}_{22} \right) \frac{\epsilon_q}{\epsilon_{q+1}} \cos \xi_q \Delta z_q - \left( \mathbf{W}_{11} - \frac{\xi_n}{k_0} \mathbf{W}_{12} \right) j \frac{\xi_q}{k_0} \sin \xi_q \Delta z_q.
\]

(A.102)
Thus,

\[
C_{q}^{m',HM} \cos \xi_{q}(z - z_{q}) + j \frac{k_{0}}{\xi_{q}} B_{q}^{m',HM} \sin \xi_{q}(z - z_{q})
\]

\[
= \left( W_{11} - \frac{\xi_{n}}{k_{0}} W_{12} \right) \left[ \cos \xi_{q} \Delta z \cos \xi_{q}(z - z_{q}) + \sin \xi_{q} \Delta z \sin \xi_{q}(z - z_{q}) \right]
\]

\[-j \frac{\epsilon_{q}}{\epsilon_{q+1}} \frac{k_{0}}{\xi_{q}} \left( W_{21} - \frac{\xi_{n}}{k_{0}} W_{22} \right) \left[ \sin \xi_{q} \Delta z \cos \xi_{q}(z - z_{q}) - \cos \xi_{q} \Delta z \sin \xi_{q}(z - z_{q}) \right]
\]

\[
= \left[ W_{11} - \xi_{n} \frac{k_{0}}{k_{0}} W_{12} - \frac{\epsilon_{q}}{\epsilon_{q+1}} \frac{k_{0}}{\xi_{q}} \left( W_{21} - \frac{\xi_{n}}{k_{0}} W_{22} \right) \right] \frac{e^{j\xi_{q}(z_{q+1} - z)}}{2}
\]

\[+ \left[ W_{11} - \xi_{n} \frac{k_{0}}{k_{0}} W_{12} + \frac{\epsilon_{q}}{\epsilon_{q+1}} \frac{k_{0}}{\xi_{q}} \left( W_{21} - \frac{\xi_{n}}{k_{0}} W_{22} \right) \right] \frac{e^{-j\xi_{q}(z_{q+1} - z)}}{2}. \quad (A.103)
\]

For large \(\lambda\),

\[
W = \begin{cases} 
\begin{bmatrix} 1 & 0 \\
0 & 1 
\end{bmatrix}; & q + 1 = n, \\
W_{11}^{\infty} \begin{bmatrix} 1 & \frac{\epsilon_{n-1}k_{0}}{\epsilon_{n}k_{0}} \\
\epsilon_{n} & \epsilon_{n} \end{bmatrix}; & \text{otherwise},
\end{cases} \quad (A.104)
\]

where

\[
W_{11}^{\infty} = \begin{cases} 
\frac{e^{\lambda \Delta z_{n-1}}}{2}; & n = q + 2 \\
\frac{e^{\lambda(z_{n} - z_{q+1})}}{2^{n-q-1}} \prod_{k=q+1}^{n-2} \left( 1 + \frac{\epsilon_{k}}{\epsilon_{k+1}} \right); & \text{otherwise},
\end{cases} \quad (A.105)
\]

thus, one obtains the following approximations

\[
W_{11} - \frac{\xi_{n}}{k_{0}} W_{12} \simeq W_{11}^{\infty} \left( 1 + \frac{\epsilon_{n-1}}{\epsilon_{n}} \right),
\]

\[
W_{21} - \frac{\xi_{n}}{k_{0}} W_{22} \simeq W_{11}^{\infty} \frac{j\lambda}{k_{0}} \left( 1 + \frac{\epsilon_{n-1}}{\epsilon_{n}} \right).
\]

Also from (A.84), it can be found that

\[
\Delta_{m, HM}^{m, HM} = U_{21}^{m, HM} - \frac{\xi_{n}}{k_{0}} U_{22}^{m, HM}
\]

\[= \left( W_{21} - \frac{\xi_{n}}{k_{0}} W_{22} \right) \epsilon_{1} \cos \xi_{1} z_{2} - \left( W_{11} - \frac{\xi_{n}}{k_{0}} W_{12} \right) j \frac{\xi_{1}}{k_{0}} \sin \xi_{1} z_{2}. \quad (A.106)
\]
Using the above results along with (A.103) yields

\[
\frac{1}{\Delta m, HM} \left[ C_{q}^m, HM \cos \xi_q (z - z_q) + j k_0 B_{q}^m, HM \sin \xi_q (z - z_q) \right] \\
\simeq -j k_0 \frac{2^{q-1}}{\lambda (1 + \frac{\epsilon_q}{\epsilon_q}) \cdots (1 + \frac{\epsilon_{q-1}}{\epsilon_q})} \left[ e^{-\lambda z} + \frac{\epsilon_{q+1} - \epsilon_q e^{-\lambda(2z_{q+1} - z)}}{\epsilon_{q+1} + \epsilon_q} \right] \\
= -j k_0 \gamma_q \left[ e^{-\lambda z} + \alpha_q e^{-\lambda(2z_{q+1} - z)} \right],
\]  

(A.107)

where \( \alpha_q = \frac{\epsilon_{q+1} - \epsilon_q}{\epsilon_{q+1} + \epsilon_q} \), and \( \gamma_q = 2^{q-1} \prod_{k=1}^{q-1} (1 + \frac{\epsilon_k}{\epsilon_{k+1}})^{-1} \). Therefore, \( f_{V_{q}^{HM}}^{\infty} \) can be given by

\[
f_{V_{q}^{HM}}^{\infty} = -\frac{\gamma_q}{\lambda k_0^2} \left[ e^{-\lambda z} + \alpha_q e^{-\lambda(2z_{q+1} - z)} \right],
\]  

(A.108)

and using the following identity

\[
\int_{0}^{\infty} J_1(\beta x) e^{-\alpha x} x dx = \frac{\beta}{(\alpha^2 + \beta^2)^{3/2}} \quad \text{Re } \alpha > |\text{Im } \beta|,
\]  

(A.109)

one obtains

\[
\int_{\lambda_m}^{\infty} f_{V_{q}^{HM}}^{\infty} J_1(\lambda \rho) \lambda^2 d\lambda = \int_{0}^{\lambda_m} f_{V_{q}^{HM}}^{\infty} J_1(\lambda \rho) \lambda^2 d\lambda - \int_{0}^{\infty} f_{V_{q}^{HM}}^{\infty} J_1(\lambda \rho) \lambda^2 d\lambda \\
= -\frac{\gamma_q}{k_0^2} \left[ \frac{\rho}{(z^2 + \rho^2)^{3/2}} + \alpha_q \frac{\rho}{((2z_{q+1} - z)^2 + \rho^2)^{3/2}} \right. \\
- \left. \int_{0}^{\lambda_m} \left( e^{-\lambda z} + \alpha_q e^{-\lambda(2z_{q+1} - z)} \right) J_1(\lambda \rho) \lambda d\lambda \right].
\]  

(A.110)

Finally, for \( S_{HM}^{1} \) and \( T_{HM}^{1} \), by using (A.84) and (A.85), it can be easily found that

\[
\frac{C_{1}^{m', HM}}{\Delta m, HM} \bigg|_{\lambda \to \infty} \simeq -j k_0 \frac{\lambda}{\lambda},
\]  

(A.111)

\[
\frac{B_{1}^{e', HM}}{\Delta e, HM} \bigg|_{\lambda \to \infty} \simeq j \lambda \frac{\lambda}{k_0}.
\]  

(A.112)

Thus,

\[
f_{S_{HM}}^{\infty} = -\frac{\epsilon_1}{\epsilon_0 k_0 \lambda},
\]  

(A.113)
and it follows that
\[
\int_{\lambda_m}^{\infty} f_{SHM}^\infty J_0(\lambda \rho) \lambda d\lambda = \int_{0}^{\infty} f_{SHM}^\infty J_0(\lambda \rho) \lambda d\lambda - \int_{0}^{\lambda_m} f_{SHM}^\infty J_0(\lambda \rho) \lambda d\lambda \\
= - \frac{j}{k_0 \epsilon_0} \left[ \frac{1}{\rho} - \int_{0}^{\lambda_m} J_0(\lambda \rho) d\lambda \right].
\] (A.114)

Likewise, \( f_{THM}^\infty \) can be given by
\[
f_{THM}^\infty = \frac{j \lambda}{k_0} + \frac{\epsilon_1 j k_0}{\epsilon_0},
\] (A.115)

and it follows that
\[
\int_{\lambda_m}^{\infty} f_{THM}^\infty J_0(\lambda \rho) \frac{1}{\lambda} d\lambda \\
= \frac{j}{k_0} \left[ \int_{0}^{\infty} J_0(\lambda \rho) d\lambda - \int_{0}^{\lambda_m} J_0(\lambda \rho) d\lambda \right] + j k_0 \frac{\epsilon_1}{\epsilon_0} \int_{\lambda_m}^{\infty} J_0(\lambda \rho) \frac{1}{\lambda^2} d\lambda \\
= \frac{j}{k_0} \left[ \frac{1}{\rho} - \int_{0}^{\lambda_m} J_0(\lambda \rho) d\lambda \right] + j k_0 \frac{\epsilon_1}{\epsilon_0} \int_{\lambda_m}^{\infty} J_0(\lambda \rho) \frac{1}{\lambda^2} d\lambda.
\] (A.116)

Since
\[
\int_{\lambda_m}^{\infty} \frac{J_0(\lambda \rho)}{\lambda^2} d\lambda = - \frac{J_0(\lambda \rho)}{\lambda} \bigg|_{\lambda_m}^{\infty} - \int_{\lambda_m}^{\infty} \rho J_1(\lambda \rho) \frac{d}{d\lambda} d\lambda \\
= \frac{J_0(\lambda_m \rho)}{\lambda_m} - \rho \int_{\lambda_m}^{\infty} \left[ \rho J_0(\lambda \rho) - \frac{d}{d\lambda} J_1(\lambda \rho) \right] d\lambda \\
= \frac{J_0(\lambda_m \rho)}{\lambda_m} - \rho J_1(\lambda_m \rho) - \rho + \int_{0}^{\lambda_m} \rho^2 J_0(\lambda \rho) d\lambda,
\] (A.117)

(A.116) becomes
\[
\int_{\lambda_m}^{\infty} f_{THM}^\infty J_0(\lambda \rho) \frac{1}{\lambda} d\lambda \\
= \frac{j}{k_0 \rho} + j k_0 \frac{\epsilon_1}{\epsilon_0} \left( \frac{J_0(\lambda_m \rho)}{\lambda_m} - \rho J_1(\lambda_m \rho) - \rho \right) - \int_{0}^{\lambda_m} \left[ \frac{1}{k_0} - \frac{\epsilon_1 k_0 \rho^2}{\epsilon_0} \right] j J_0(\lambda \rho) d\lambda.
\] (A.118)
A.5 Derivation of (3.10)

Consider the integral

\[ I = \int_0^{\pi} \frac{d\gamma}{\gamma - \gamma_p} \]  (A.119)

where \( \gamma_p \) denotes a pole, which can be either surface or leaky wave pole, and the integration path is shown in figure 3.1. Let \( \gamma_p = \alpha_p + j \beta_p \) and \( \gamma_l = \pi/2 + j \delta_l \), and divide the integral \( I \) into two parts as follows:

\[ I = I_1 + I_2. \]  (A.120)

Notice that the integration path for \( I_1 \) is on the real axis and \( \frac{1}{\gamma - \gamma_p} \) on the real axis can be rewritten as

\[ \frac{1}{\gamma - \gamma_p} = \frac{1}{\gamma - \alpha_p - j \beta_p} = \frac{1}{\gamma - \alpha_p + j \beta_p} \]  (A.121)

\( I_1 \) becomes

\[ I_1 = \int_0^{\pi/2} \frac{\gamma - \alpha_p}{(\gamma - \alpha_p)^2 + \beta_p^2} d\gamma + j \int_0^{\pi/2} \frac{\beta_p}{(\gamma - \alpha_p)^2 + \beta_p^2} d\gamma \]

\[ = \frac{1}{2} \ln[(\gamma - \alpha_p)^2 + \beta_p^2] \bigg|_0^{\pi/2} + j \tan^{-1}\left(\frac{\gamma - \alpha_p}{\beta_p}\right) \bigg|_0^{\pi/2} \]

\[ = \frac{1}{2} \ln \left(\frac{\pi/2 - \alpha_p}{\alpha_p^2 + \beta_p^2} \right) + j \tan^{-1}\left(\frac{\pi/2 - \alpha_p}{\beta_p}\right) - j \tan^{-1}\left(\frac{-\alpha_p}{\beta_p}\right). \]  (A.122)

Likewise, consider the integration path for \( I_2 \), it can be rewritten by using the change of variable \( \gamma = \pi/2 + j \delta \) as

\[ I_2 = j \int_0^{\delta_i} \frac{d\delta}{\gamma - \gamma_p}, \]  (A.123)
where
\[
\frac{1}{\gamma - \gamma_p} = \frac{1}{\pi/2 + j\delta - \alpha_p - j\beta_p} = \frac{1}{\pi/2 - \alpha_p - j(\delta - \beta_p)} - \frac{j\beta_p}{(\pi/2 - \alpha_p)^2 + (\delta - \beta_p)^2}.
\] (A.124)

Thus, \(I_2\) becomes
\[
I_2 = j \int_0^{\delta_i} \frac{\pi/2 - \alpha_p}{(\pi/2 - \alpha_p)^2 + (\delta - \beta_p)^2} d\delta + \int_0^{\delta_i} \frac{\delta - \beta_p}{(\pi/2 - \alpha_p)^2 + (\delta - \beta_p)^2} d\delta
= j \tan^{-1}\left(\frac{\delta - \beta_p}{\pi/2 - \alpha_p}\right) \bigg|_0^{\delta_i} + \frac{1}{2} \ln[(\pi/2 - \alpha_p)^2 + (\delta - \beta_p)^2] \bigg|_0^{\delta_i}
= j \tan^{-1}\left(\frac{\delta_i - \beta_p}{\pi/2 - \alpha_p}\right) - j \tan^{-1}\left(\frac{-\beta_p}{\pi/2 - \alpha_p}\right) + \frac{1}{2} \ln\frac{(\pi/2 - \alpha_p)^2 + (\delta_i - \beta_p)^2}{(\pi/2 - \alpha_p)^2 + \beta_p^2}.
\] (A.125)

From (A.122) and (A.125), the real part of \(I\) becomes
\[
\text{Re}I = \frac{1}{2} \ln\frac{(\alpha_p - \pi/2)^2 + (\beta_p - \delta_i)^2}{\alpha_p^2 + \beta_p^2}.
\] (A.126)

To obtain the imaginary part, the locations of poles must be taken into account. For surface wave poles, \(\alpha_p \geq \pi/2\) and \(\delta_i > \beta_p > 0\), while \(\alpha_p < \pi/2\) and \(\beta_p < 0\) for leaky wave poles. Now, using the following inverse trigonometric identities:
\[
\tan^{-1}(-x) = -\tan^{-1}x,
\] (A.127)
and
\[
\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} 
-\frac{\pi}{2} - \tan^{-1}x & \text{for } x < 0 \\
\frac{\pi}{2} - \tan^{-1}x & \text{for } x > 0,
\end{cases}
\] (A.128)
then for surface wave poles, the imaginary part of \( I \) becomes

\[
\text{Im} I = \tan^{-1}\left( \frac{\pi/2 - \alpha_p}{\beta_p} \right) - \tan^{-1}\left( \frac{-\alpha_p}{\beta_p} \right) + \tan^{-1}\left( \frac{\delta_l - \beta_p}{\pi/2 - \alpha_p} \right) - \tan^{-1}\left( \frac{-\beta_p}{\pi/2 - \alpha_p} \right) \\
= \tan^{-1}\left( \frac{\pi/2 - \alpha_p}{\beta_p} \right) + \tan^{-1}\left( \frac{\alpha_p}{\beta_p} \right) - \frac{\pi}{2} - \tan^{-1}\left( \frac{\pi/2 - \alpha_p}{\delta_l - \beta_p} \right) \\
- \frac{\pi}{2} - \tan^{-1}\left( \frac{\pi/2 - \alpha_p}{\beta_p} \right) \\
= -\pi + \tan^{-1}\left( \frac{\alpha_\beta - \pi/2}{\delta_l - \beta_p} \right) + \tan^{-1}\left( \frac{\alpha_p}{\beta_p} \right). \quad \text{(A.129)}
\]

Likewise, the imaginary part of \( I \) for leaky wave poles becomes

\[
\text{Im} I = \tan^{-1}\left( \frac{\pi/2 - \alpha_p}{\beta_p} \right) - \tan^{-1}\left( \frac{-\alpha_p}{\beta_p} \right) + \tan^{-1}\left( \frac{\delta_l - \beta_p}{\pi/2 - \alpha_p} \right) - \tan^{-1}\left( \frac{-\beta_p}{\pi/2 - \alpha_p} \right) \\
= \tan^{-1}\left( \frac{\pi/2 - \alpha_p}{\beta_p} \right) - \tan^{-1}\left( \frac{\alpha_p}{\beta_p} \right) + \frac{\pi}{2} - \tan^{-1}\left( \frac{\pi/2 - \alpha_p}{\delta_l - \beta_p} \right) \\
- \frac{\pi}{2} - \tan^{-1}\left( \frac{\pi/2 - \alpha_p}{\beta_p} \right) \\
= -\tan^{-1}\left( \frac{\pi/2 - \alpha_p}{\delta_l - \beta_p} \right) - \tan^{-1}\left( \frac{\alpha_p}{\beta_p} \right). \quad \text{(A.130)}
\]

(3.10) can then be obtained by combining results given in (A.126), (A.129), and (A.130).

For a lossless medium, the real part of any surface wave pole becomes \( \pi/2 \), i.e., \( \gamma_p = \pi/2 + j\beta_p \), thus the integration path for \( I_2 \) will pass through the pole. One can deform the integration path to be the one shown in figure A.1. Since

\[
\frac{1}{\gamma - \gamma_p} = \frac{1}{\pi/2 + j\delta - \pi/2 - j\beta_p} \\
= -\frac{j}{\delta - \beta_p}, \quad \text{(A.131)}
\]

\( I_2 \) becomes

\[
I_2 = \int_{C_1} \frac{d\delta}{\delta - \beta_p} + \int_{C_2} \frac{d\delta}{\delta - \beta_p} + \int_{C_3} \frac{d\delta}{\delta - \beta_p}. \quad \text{(A.132)}
\]
It follows that

\[
\int_{C_1} \frac{d\delta}{\delta - \beta_p} = \int_0^{\beta_p - \epsilon} \frac{d\delta}{\delta - \beta_p} = \ln |\epsilon| - \ln |\beta_p|, \tag{A.133}
\]

and

\[
\int_{C_3} \frac{d\delta}{\delta - \beta_p} = \int_{\beta_p + \epsilon}^{\delta_{l}} \frac{d\delta}{\delta - \beta_p} = \ln |\delta_{l} - \beta_p| - \ln |\epsilon|. \tag{A.134}
\]

For the integral along \(C_2\), using the change of variable \(\delta = \beta_p + \epsilon e^{-j\phi}\) for \(\phi \in [\pi/2, 3\pi/2]\) yields

\[
\int_{C_2} \frac{d\delta}{\delta - \beta_p} = \int_{\pi/2}^{3\pi/2} \frac{1}{\beta_p + \epsilon e^{-j\phi} - \beta_p} e^{-j\phi} d\phi
= -j \int_{\pi/2}^{3\pi/2} d\phi
= -j\pi. \tag{A.135}
\]

Combining the results from (A.133),(A.134) and (A.135), along with (A.122), one obtains

\[
\text{Re} I = \frac{1}{2} \ln \frac{(\beta_p - \delta_{l})^2}{(\pi/2)^2 + \beta_p^2}, \tag{A.136}
\]

and

\[
\text{Im} I = -\pi + \tan^{-1}\left(\frac{\pi}{2\beta_p}\right), \tag{A.137}
\]

which is exactly the result when substituting \(\alpha_p = \pi/2\) into (A.126) and (A.129).
Figure A.1: A deformed integration path for the case of lossless medium.
APPENDIX B

POSTPROCESSING

Once the MoM matrix equation is solved and the unknown currents are obtained, they can be used to compute various quantities that characterize antenna arrays, such as radiation pattern, radar cross section, or input impedance. This appendix will first discuss how to calculate the far-zone scattered field from the equivalent currents, which is used to determine the far-field quantities of antenna arrays, namely, radiation pattern for radiation problems and radar cross section for scattering problems. It will also present how to compute the circuit quantities of antenna arrays, such as the input impedance or the port impedance matrix. These parameters are used in the analysis of complete antenna array systems that include both arrays and feed networks.

B.1 Scattered Field

Using the asymptotic evaluation of the Sommerfeld integral, the far-zone electric field in the free space region radiated by a horizontal electric point source can be given by [36]:

$$E_f = \frac{j \omega \mu_m \cos \theta}{2\pi r} e^{-jkr} \left[ \frac{C_{n,m}^{*,HE}}{\Delta_{e,HE}} \hat{\phi} \hat{\rho} - \frac{\mu_0}{\mu_m} \frac{C_{n,m}^{*,HE}}{\Delta_{m,HE}} \hat{\theta} \hat{\rho} \right] \cdot \mathbf{p}_t, \quad \text{(B.1)}$$

where

$$\hat{\rho} = \hat{x} \cos \phi + \hat{y} \sin \phi, \quad \text{(B.2)}$$
\[ \hat{\phi} = -\hat{x} \sin \phi + \hat{y} \cos \phi, \quad (B.3) \]
\[ \hat{\theta} = \hat{x} \cos \theta \cos \phi + \hat{y} \cos \theta \sin \phi - \hat{z} \sin \theta, \quad (B.4) \]

\( p_t \) denotes a horizontal electric point source, and \( C_{m',HE} \), \( C_{e',HE} \), \( \Delta_{m,HE} \), \( \Delta_{e,HE} \) are defined in section 2.2. Therefore, the far-zone electric field radiated by a patch mode current given in (4.8) and (4.9) can be written as:

\[
E_f = j\mu_m \cos \theta \frac{e^{-jk_0r}}{r} \frac{k_e h^2}{\sin k_e h} \sin \frac{k_0 w}{2} \sin \theta \sin (\phi - \phi') \cos [k_0 h \sin \theta \cos (\phi - \phi')] - \frac{\mu_0}{\mu_m} \frac{C_{m',HE}}{\Delta_{m,HE}} \sin (\phi - \phi') \hat{\phi} - \frac{\mu_0}{\mu_m} \frac{C_{e',HE}}{\Delta_{e,HE}} \cos (\phi - \phi') \hat{\theta}, \quad (B.5)
\]

where

\[ \phi' = \begin{cases} 0 & \text{for } \hat{x} \text{-directed patch mode} \\ \frac{\pi}{2} & \text{for } \hat{y} \text{-directed patch mode} \end{cases}, \quad (B.6) \]

and \( h, w \) denote the half-length and the width, respectively, of the patch mode. It is also noted that the phase reference is at the center of the patch mode. The far-zone scattered field can then be found by summing up the fields radiated from all array elements or all patch modes as

\[
E^s = \sum_{n=1}^{N_x} i_n^x E_f(f_n^x) + \sum_{m=1}^{N_y} i_m^y E_f(f_m^y), \quad (B.7)
\]

where \( E_f(f) \) denotes the far-zone electric field radiated by mode \( f \), and \( i_n^x, f_n^x, i_m^y, f_m^y \) represent the \( n^{th} \) \( \hat{x} \)-directed and \( m^{th} \) \( \hat{y} \)-directed patch mode currents, respectively.

For the radiation problem, a radiation pattern can be obtained by just simply calculating the far-zone scattered field of an antenna array with the term \( e^{-jk_0r}/r \) excluded. It is typically presented in two components, namely, \( E_{\theta} \) and \( E_{\phi} \), to observe the co-polarization and cross-polarization components. It is also often given in the [dBi] unit, which represents the total electric field from an array with respect to the electric field due to the isotropic antenna.
with the same radiated power. It is given by

\[ E_\zeta \text{ [dBi]} = 20 \log_{10} \left( \frac{4\pi |E_\zeta|^2}{2\eta P_{rad}} \right), \]  

(B.8)

where \( \eta \) denotes the intrinsic impedance, \( P_{rad} \) denotes the radiated power, and \( \zeta \in \{\theta, \phi\} \).

For the scattering problem, the radar cross section (RCS) is typically calculated to characterize the scattering properties of antenna arrays. The RCS, denoted here by \( \sigma_{\zeta\xi} \), is defined as

\[ \sigma_{\zeta\xi} = 4\pi r^2 \frac{|E_\zeta|^2}{|E_{\xi \text{inc}}|^2}, \]  

(B.9)

where

- \( \sigma_{\zeta\xi} \) the \( \zeta \) component of the RCS due to the \( \xi \) polarized incident field
- \( \zeta \in \{\theta, \phi\} \) the component of the RCS, either \( \theta \) or \( \phi \)
- \( \xi \in \{\theta, \phi\} \) the polarization of the incident field, either TE(perpendicular or \( \phi \)) or TM(parallel or \( \theta \))
- \( E_\zeta \) The \( \zeta \) component of the scattered field
- \( E_{\xi \text{inc}} \) The \( \xi \) polarization of the incident field.

In general, RCS is presented in terms of area, and square meter (sm) is the most common unit. Furthermore, the dB scale is typically used and the common unit for RCS becomes [dBsm].

### B.2 Input Impedance and Port Impedance Matrix

Consider an antenna as a single port device, the input impedance can be defined as the ratio of the voltage to the current at the antenna port, i.e.,

\[ Z_{in} = \frac{V}{I}. \]  

(B.10)

If the basis functions are normalized to be 1 A, the MoM solutions simply represent the peak currents of the expansion modes, which makes it convenient to compute the input impedance. This approach can be easily extended to define the port impedance matrix for
an array when considering an array as a $N$-port device, where $N$ is equal to the number of array elements. The $(ij)^{th}$ element of this matrix is defined as

$$Z_{ij} = \left. \frac{V_i}{I_j} \right|_{I_k=0 \text{ for } k \neq j}, \quad \text{(B.11)}$$

where $V_i$ is the voltage at the $i^{th}$ port and $I_j$ is the current at the $j^{th}$ port. It is equal to the ratio of the induced voltage at the $i^{th}$ port to the current at the $j^{th}$ port, when only the $j^{th}$ port is excited. The next subsections will discuss how to calculate input impedance and port impedance matrix for arrays of different elements.

### B.2.1 Strip Dipole Array

Since the delta gap generator is used to model the excitation of a strip dipole antenna, for a single dipole, the voltage at the input port is simply equal to the driving voltage, thus the input impedance is given by

$$Z_{\text{in}} = \frac{V}{i_k}, \quad \text{(B.12)}$$

where $i_k$ denotes the coefficient of the $k^{th}$ expansion mode, which is the center mode of the dipole, and $V$ denotes the driving voltage.

For the array case, since the source used here is the voltage source, it is more convenient to obtain the port admittance matrix, whose $(k, l)^{th}$ element is given by

$$Y_{kl} = \left. \frac{i_k}{V_l} \right|_{V_m=0 \text{ for } m \neq l}, \quad \text{(B.13)}$$

where $i_k$ is the coefficient of the center mode of the $k^{th}$ element and $V_l$ is the driving voltage for the $l^{th}$ element. It is noted that if only one expansion mode is used, the port admittance matrix, denoted here by $Y^p$, simply equals to the inverse matrix of the MoM operator matrix, i.e.,

$$Y^p = Z^{-1}. \quad \text{(B.14)}$$
B.2.2 Microstrip-line Fed Patch Antenna Array

As mentioned earlier, the excitation of the microstrip-line fed patch antenna is modelled using the equivalent vertical current ribbon which is proportional to the driving current. Let $I^i$ be the driving current, then the input impedance can be given by [11]

\[
Z_{in} = \frac{-\int_V \mathbf{E}^s \cdot \mathbf{J}^i dv}{(I^i)^2} = \frac{-\sum_m i_m \int_V \mathbf{E}^s(f_m) \cdot \mathbf{J}^i dv}{(I^i)^2}
\] (B.15)

where $\mathbf{E}^s$ denotes the electric field from all expansion modes of the patch, $\mathbf{E}^s(f_m)$ denotes the electric field radiated by the $m^{th}$ expansion mode, $i_m$ is the coefficient associated with the $m^{th}$ expansion mode, and $\mathbf{J}^i$ denotes the vertical current ribbon. Since the $m^{th}$ element of the excitation vector is given by

\[
v_m = \int_S \mathbf{E}(\mathbf{J}^i) \cdot f_m ds,
\] (B.16)

where $\mathbf{E}(\mathbf{J}^i)$ denotes the electric field radiated by $\mathbf{J}^i$ and $f_m$ denotes the $m^{th}$ expansion function. By reciprocity, $v_m$ can be rewritten as

\[
v_m = \int_V \mathbf{E}^s(f_m) \cdot \mathbf{J}^i dv.
\] (B.17)

Assume that the driving current $I^i = 1A$, then the input impedance simply becomes

\[
Z_{in} = -i^T v.
\] (B.18)

where $i$, $v$ denote the current vector and excitation vector, respectively.

Likewise, the port impedance matrix, $\mathbf{Z}^p$, can be defined as [30],

\[
\mathbf{Z}^p i^p = v^p,
\] (B.19)
where $\mathbf{c}^p$, $\mathbf{v}^p$ denote the port current vector and port voltage vector, respectively. Recall that the MoM matrix equation is given by

$$
Z \mathbf{i} = \mathbf{v},
$$
(B.20)

and $v_k$ is proportional to the driving current at the $k$ element, i.e., $\mathbf{v}$ is related to $\mathbf{c}^p$. Let the $(m, k)^{th}$ mode denote the $m^{th}$ mode of the $k^{th}$ element, then the $(m, k)^{th}$ element of $\mathbf{v}$ is given by

$$
v_{m,k} = \int_S E_k(J_k^i) \cdot \mathbf{f}_{m,k} ds
= i_k^p \int_S E_k(I_z) \cdot \mathbf{f}_{m,k} ds,
$$
(B.21)

where $J_k^i$ the excited current source at the $k^{th}$ element
$E_k(J_k^i)$ the electric field due to $J_k^i$
$I_z$ the unit vertical current
$\mathbf{f}_{m,k}$ the $m^{th}$ mode of the $k^{th}$ element
$i_k^p$ the driving current of the $k^{th}$ element.

It is noted that $I_z = \hat{z} \delta(x - x_p) \delta(y - y_p)$ for $0 \leq z \leq d$, $d$ is the thickness of the substrate, and $(x_p, y_p)$ denotes the location of the feed point. Assume that the feed point is the same for all patch antennas, then $v_{m,k} = \mathbf{v}^{mode}$, for all $k, k = 1, \ldots, N$, where $N$ is the number of elements. Now, let $\mathbf{v}^{mode} = [v_1^{mode} v_2^{mode} \cdots v_N^{mode}]^T$, then $\mathbf{v}$ can be related to $\mathbf{c}^p$ as

$$
\mathbf{v} = \mathbf{V}^{mode} \mathbf{c}^p,
$$
(B.22)

where

$$
\mathbf{V}^{mode} = \mathbf{I}_N \otimes \mathbf{v}^{mode},
$$
(B.23)

$\mathbf{I}_N$ denotes the $N \times N$ identity matrix, and the operator $\otimes$ denotes the Kronecker product.

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Likewise, the $k^{th}$ element of $\mathbf{v}^p$ is given by

\[
v^p_k = - \int_0^d \hat{z} \cdot \mathbf{E}^s_k dz
\]

\[
= - \sum_m i_{m,k} \int_0^d \hat{z} \cdot \mathbf{E}^s_k(f_{m,k}) dz
\]

\[
= - \sum_m i_{m,k} \int_V \mathbf{E}^s_k(f_{m,k}) \cdot I_z dv
\]

(B.24)

where $\mathbf{E}^s_k$ denotes the electric field due to the $k^{th}$ element, and $\mathbf{E}^s_k(f_{m,k})$ denotes the electric field due to $f_{m,k}$. Since by reciprocity,

\[
\int_V \mathbf{E}^s_k(f_{m,k}) \cdot I_z dv = \int_S \mathbf{E}^s_k(J^i_k) \cdot f_{m,k} ds,
\]

(B.25)

(B.24) becomes

\[
v^p_k = - \sum_m i_{m,k} \int_S \mathbf{E}^s_k(I_z) \cdot f_{m,k} ds
\]

\[
= - \sum_m i_{m,k} v^{mode}_m.
\]

(B.26)

where $J^i_k = i^p_k I_z$ as before. Therefore, $\mathbf{v}^p$ can be related to $\hat{z}$ as

\[
\mathbf{v}^p = - (\mathbf{V}^{mode})^T \hat{z}.
\]

(B.27)

It is also noted that $\hat{z}$ can be obtained from (B.20) as $Z^{-1} \mathbf{v}$, by using (B.22) and (B.27), (B.19) can be rewritten as

\[
Z^p \hat{z}^p = \mathbf{v}^p
\]

\[
= - (\mathbf{V}^{mode})^T \hat{z}
\]

\[
= - (\mathbf{V}^{mode})^T Z^{-1} \mathbf{v}
\]

\[
= - (\mathbf{V}^{mode})^T Z^{-1} \mathbf{V}^{mode} \hat{z}^p.
\]

(B.28)

Therefore, $Z^p$ can be found to be

\[
Z^p = - (\mathbf{V}^{mode})^T Z^{-1} \mathbf{V}^{mode}.
\]

(B.29)
Clearly, when only one mode is used, $v^{\text{mode}}$ becomes a scalar and $V^{\text{mode}}$ becomes a diagonal matrix. Let $v^{\text{mode}} = \int_S \mathbf{E}(\mathbf{J}_k) \cdot \mathbf{f}_k \, ds$ for all $k, k = 1, \ldots, N$, where $\mathbf{f}_k$ denotes the expansion function for the $k^{th}$ element, then $V^{\text{mode}} = v^{\text{mode}} \mathbf{I}$ and (B.29) can be simplified to be

$$Z^p = - (v^{\text{mode}})^2 Z^{-1}. \quad (B.30)$$

### B.2.3 Probe-fed Patch Array

Since the excitation model used for the probe-fed patch is either the magnetic frill generator or voltage gap generator, the voltage at the terminal is known. Since the MoM solutions in this case will also give the currents on the wire and thus the current at the input port can be obtained. For the single patch case, the input impedance can be simply found from

$$Z_{\text{in}} = \frac{V^i}{i^w_1} \quad (B.31)$$

where $V^i$ is the driving voltage and $i^w_1$ is the coefficient of the first wire mode.

For the probe-fed array case, it is more convenient to compute the port admittance matrix, denoted here by $Y^p$, instead of $Z^p$. The $(k, l)^{th}$ element of $Y^p$ can be found from

$$Y^{p}_{kl} = \left. \frac{i^w_{k,1} V^i_l}{V^i_l} \right|_{V_m=0 \text{ for } m \neq l}, \quad (B.32)$$

where $i^w_{k,1}$ is the first wire mode of the $k^{th}$ element and $V^i_l$ is the driving voltage for the $l^{th}$ element. It is equal to the ratio of the induced current at the $k^{th}$ port to the driving voltage at the $l^{th}$ port when only the $l^{th}$ port is closed.

As can be expected from discussions in section B.2.1-B.2.3, regardless of the element type, the calculation of the port impedance matrix, or port admittance matrix, will involve either the inversion of the MoM operator matrix equation or solving MoM matrix equation $N$ times for $N$ array element, thus the computational cost will become quite expensive.
APPENDIX C

FEED NETWORK ANALYSIS

An antenna array system typically consists of an antenna array and a feed network, which is used to either supply or receive power. In general, the more the number of elements in an antenna array, the larger the size of corresponding feed network and the more computational resources required for solving this problem using full-wave or MoM analysis. Furthermore, since feed networks have to be designed such that they can operate well with corresponding antenna arrays in order to achieve desirable performance, in most cases their design process may require many iterations, which can result in a highly inefficient and time-consuming process and can be intractable for large antenna arrays. One approach to improve the efficiency of the analysis of such problems is to decompose the problem into two major parts, one dealing with antenna arrays alone and the other dealing with feed networks alone, and then apply a generalized Thevenin’s theorem to systematically combine the two parts together. Discussed in this chapter is a generalized Thevenin’s theorem which can be used to combine a result from an antenna array full-wave analysis with a feed network.

Consider the array configuration as shown in figure C.1, which consists of a current source/generator value $I_g$ with source admittance $Y_g$, an adjustable load admittance, $Y_L$, a feed network and N-element antenna array. It can be viewed as an N+1-port network with
one current source. The matrix relating the port voltages, $V_i, i = 0, \cdots, N$, to the port currents, $I_i, i = 0, \cdots, N$, at the input and output ports of the feed network is given by the relation [71]:

$$[V] = [Z_{oc}][I]$$

(C.1)

where

$$[V] = \begin{bmatrix} V_0 \\ V_n \end{bmatrix}$$

(C.2)

$$[V_n] = [V_1 V_2 \cdots V_N]^T,$$

(C.3)

$$[Z_{oc}] = \begin{bmatrix} Z_{00} & [Z_{0m}] \\ [Z_{n0}] & [Z_{nm}] \end{bmatrix}$$

(C.4)

$$[I] = \begin{bmatrix} I_0 \\ I_m \end{bmatrix}$$

(C.5)

$$[I_m] = [I_1 I_2 \cdots I_N]^T,$$

(C.6)

Note that the $0^{th}$ port denotes that which connects to the source, whereas the $1^{st}, \cdots, N^{th}$ ports denote those that connect to the array elements, i.e., to the output ports. The open-circuit impedance matrix $Z_{oc}$ can be obtained from the feed network design.

![Array-feed network configuration](image)

Figure C.1: An array-feed network configuration
At the output ports, $[V_n]$ can be related to $[I_m]$ by the following equation:

$$[V_n] = -[Z_{nm}^a][I_m] \tag{C.7}$$

where $[Z_{nm}^a]$ represents the port impedances of the antenna array. If there are point phase shifters, (C.7) becomes:

$$[V_n] = -[Z_{nm}^a][P][I_m] \tag{C.8}$$

where the phase shifter matrix $[P]$ has the elements:

$$P_{nm} = e^{-j\beta l_{nm}}\delta_{nm}; \quad \beta l_{nm} \text{ is known,} \tag{C.9}$$

with $\delta_{nm}$ denoting the Kronecker delta. It is assumed here that the phase shifters are ideal, i.e., they are lossless and “only” transform the phase of $[I_m]$ which feed the array via the operation $[P][I_m]$.

If one defines the source voltage vector, $[V_g]$, and the load matrix $[Z_L]$, respectively, to be:

$$[V_g] = \begin{bmatrix} V_g \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad [Z_L] = \begin{bmatrix} (Y_g + Y_L)^{-1} & 0 & \cdots & 0 \\ 0 & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & [Z_{nm}^{aph}] \\ 0 & \cdots & \cdots & \cdots \end{bmatrix}, \tag{C.10}$$

where $V_g$ is the equivalent source voltage, and $[Z_{nm}^{aph}]$ is defined to be:

$$[Z_{nm}^{aph}] = [Z_{nm}^a][P]. \tag{C.12}$$

Next, the relation between voltages and currents is invoked, namely

$$[V_g] = [V] + [Z_L][I]. \tag{C.13}$$
Then using (C.1) in (C.13), one obtains:

$$[V_g] = ([Z_{oc}] + [Z_L]) [I],$$

(C.14)

which can be expanded as:

$$( [Z^f_{nm}] + [Z_{a^n{h}}] ) [I_m] + [Z^f_{0m}] I_0 = 0,$$

(C.15)

and

$$[Z^f_{nm}] [I_m] + [Z^f_{0n}] I_0 + [Z_{a^n{h}}] [I_m] = 0.$$  \hspace{1cm} \text{(C.16)}

Also, since

$$V_0 = Z^f_{00} I_0 + [Z^f_{0m}] [I_m]$$

$$= \frac{I_g - I_0}{Y_g + Y_L},$$

(C.17)

$I_0$ can be expressed as:

$$I_0 = \frac{Y^f_{00} I_g}{Y^f_{00} + Y_g + Y_L} - \frac{Y^f_{00} Y^f_{00} [Z^f_{0m}] [I_m]}{Y^f_{00} + Y_g + Y_L},$$

(C.18)

where $Y^{f}_{00} = 1/Z^{f}_{00}$. Using (C.18) in (C.15) yields:

$$-( [Z^{inf}_{nm}] + [Z^{a^n{h}}] ) [I_m] = [V^{oc}_{n}],$$

(C.19)

where

$$[Z^{inf}_{nm}] = [Z^f_{nm}] - \frac{[Z^f_{0n}] [Z^f_{0m}]}{Z^f_{00} + (Y_g + Y_L)^{-1}},$$

(C.20)

and $[V^{oc}_{n}]$ is the open circuit voltage vector which can be shown to be:

$$[V^{oc}_{n}] = \frac{I_g Y^f_{00}}{Y_g + Y_L + Y^f_{00} [Z^f_{0n}]},$$

(C.21)

Consequently, $[I_m]$ is related to $[V^{oc}_{n}]$ by:

$$[I_m] = -( [Z^{a^n{h}}] + [Z^{inf}_{nm}] )^{-1} [V^{oc}_{n}];$$

(C.22)
the preceding result can be used to design the feed network such that the desired feed currents are achieved.

This method can also be used in the receiving problem. In this case, the $0^{th}$ port becomes the output port, while the $1^{st}, \ldots, N^{th}$ ports become the sources. One can use the voltage induced at each antenna port as the sources to the network and find the received current at the output port.

It is worthwhile noting that a generalized Norton’s theorem can be obtained in the same way by using the short-circuit admittance matrix instead of the open-circuit impedance matrix and vice versa. This can be used to connect a probe-fed patch antenna array with a feed network.
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