EM CHARACTERIZATION OF MAGNETIC PHOTONIC/DEGENERATE BAND EDGE CRYSTALS & RELATED ANTENNA REALIZATIONS

DISSERTATION

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Extraordinary properties found in engineered metamaterials have drawn great interest as they can address industry demands for small, light-weight, and multifunctional devices. It is not therefore surprising that a variety of artificial materials are being widely considered for various radio frequency (RF) applications. Among these metamaterials, a recently introduced class of anisotropic photonic crystals, namely magnetic photonic (MPC) and degenerate band edge (DBE) crystals, has been shown to exhibit unique propagation modes as compared to regular periodic assemblies. For the first time, this dissertation carries out computational and experimental analysis of these new crystals for specific RF applications. Our ultimate goal is to develop high gain antenna apertures and miniature footprint antennas. In this context, this dissertation begins by establishing an understanding of the fundamental electromagnetic properties of 1D DBE and MPC crystals using the transfer matrix and spectral domain method of moments (MoM) computations. This is followed by numerical characterization of 3D DBE crystals via surface integral equations. Upon successful demonstration of the DBE mode for improved antenna performance, a measurement setup is presented to characterize low loss uniaxial materials. Subsequently, Specifically, a finite DBE assembly is built and shown to exhibit large aperture efficiency for conformal high gain antenna applications.
The second half of the dissertation introduces a novel coupled transmission line concept capable of emulating DBE mode on otherwise uniform microwave substrates. Using this novel dual transmission line concept, we present examples of several small antennas on low and high contrast substrates, and fabricate a prototype to experimentally verify the printed slow wave concepts. The measured DBE antenna is shown to perform better than other recently published metamaterial antennas. It is therefore very attractive for several RF applications requiring small and efficient antennas (such as RF identification (RFID) tags or mobile communications). The last chapter presents a lumped circuit DBE model and suggest improvement to existing DBE antennas by introducing lumped elements into the transmission lines.
Dedicated to my brother, parents, and grandparents...
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# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Chapters</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Abstract</td>
<td>ii</td>
</tr>
<tr>
<td>Dedication</td>
<td>iv</td>
</tr>
<tr>
<td>Acknowledgments</td>
<td>v</td>
</tr>
<tr>
<td>Vita</td>
<td>vi</td>
</tr>
<tr>
<td>List of Tables</td>
<td>xiii</td>
</tr>
<tr>
<td>List of Figures</td>
<td>xiv</td>
</tr>
<tr>
<td>1. Introduction</td>
<td>1</td>
</tr>
<tr>
<td>2. RF Propagation within 1D Magnetic Photonic Crystals</td>
<td>8</td>
</tr>
<tr>
<td>2.1 Characteristics of MPCs</td>
<td>9</td>
</tr>
<tr>
<td>2.2 Analysis of 1D Finite Thickness MPCs</td>
<td>13</td>
</tr>
<tr>
<td>2.3 Propagation in Lossless MPC</td>
<td>15</td>
</tr>
<tr>
<td>2.4 Material Loss Effects</td>
<td>23</td>
</tr>
<tr>
<td>2.5 Remarks</td>
<td>25</td>
</tr>
<tr>
<td>3. Small Antennas Embedded within 1D Magnetic Photonic &amp; Degenerate Band Edge Crystals</td>
<td>26</td>
</tr>
<tr>
<td>3.1 Transmission Characteristics of Finite Thickness MPCs</td>
<td>28</td>
</tr>
</tbody>
</table>
3.2 Analysis of Strip Dipoles Embedded within 1D Finite Thickness Layered Media ......................................................... 30
3.3 Dipole Performance within the MPC ........................................... 35
3.4 MPC Parameter Effect on Dipole Performance ................................. 38
3.5 Resonance and Amplitude Increase within Degenerate Band Edge Crystals .................................................. 40
3.6 Dipole Performance in DBE Crystal ............................................ 42
3.7 Remarks ........................................................................ 46

4. MoM-SIE Solutions for Modeling 3D Uniaxial Materials Using Closed Form Dyadic Green’s Functions ........................................ 48
   4.1 PMCHWT–SIE Formulation .................................................... 50
   4.2 Uniaxial Dyadic Green’s Functions ........................................ 52
   4.3 MoM Implementation Using Curvilinear Geometry Modeling .......... 54
   4.4 Singularities of the Uniaxial Dyadic Green’s Functions ................. 56
   4.5 Numerical Examples .......................................................... 58
     4.5.1 Uniaxial Dielectric Cube .............................................. 58
     4.5.2 Uniaxial Sphere ........................................................ 58
     4.5.3 Dipole Radiation in High Contrast Uniaxial Slab ............... 60
     4.5.4 Dipole Radiation within 3D DBE Crystals ....................... 62
   4.6 Remarks ........................................................................ 64

5. Characterization of Natural and Engineered Low Loss Uniaxial Dielectric Materials at Microwave Frequencies .................. 66
   5.1 Measurement Methodology and Rutile Characterization ............. 67
     5.1.1 Cavity Design and Measurement Procedure ...................... 69
   5.2 Loss Characterization of Periodic Assemblies ............................ 77
   5.3 Remarks ........................................................................ 81

   6.1 Printed Antenna Miniaturization Using DBE Dispersion ............... 83
   6.2 DBE Unit Cell Using Printed Coupled Lines .............................. 84
   6.3 MS-DBE Antenna Performance ............................................. 87
   6.4 Miniature MS-DBE Antenna on Low-Loss Alumina Substrate ........ 91
   6.5 Remarks ........................................................................ 96
7. Lumped Circuit Models for Degenerate Band Edge and Magnetic Photonic Crystal Dispersion Diagrams ........................................... 98

7.1 Lumped Element Circuit Model of Printed DBE .................. 99
7.2 Printed DBE Design Using Dual Transmission Lines
    Loaded with Lumped Circuit Elements ............................... 102
    7.2.1 DBE Circuit Example .................................. 102
    7.2.2 DBE Antenna Example ................................ 103
7.3 Triple Coupled Lines for SIP Realization ....................... 106
7.4 Remarks ..................................................... 107

8. Conclusions ......................................................... 110

Appendices:

A. Transfer Matrix of a 1D Homogenous Layer with Anisotropic Material Parameters ......................................................... 114

B. Spectral Domain Green’s Function of Uniform Media ......... 118

C. Finite Element Method For Determining Cavity Resonances .... 121

D. Transfer Matrices of Lumped Element DBE/MPC Circuit Models ..... 123
    D.1 2-port Transmission Line Circuit Model ..................... 123
    D.2 Uncoupled Dual Transmission Line Circuit Model .......... 124
    D.3 Coupled Dual Transmission Line Circuit Model .......... 125
    D.4 Partially Coupled Triple Line Circuit Model ............ 131

Bibliography ............................................................. 136
# LIST OF TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>Propagation characteristics of lossless MPCs</td>
<td>21</td>
</tr>
<tr>
<td>2.2</td>
<td>Effect of loss on propagation characteristics</td>
<td>24</td>
</tr>
<tr>
<td>4.1</td>
<td>Computational Statistics of PMCHWT-SIE and FEM Modeling</td>
<td>63</td>
</tr>
<tr>
<td>5.1</td>
<td>FEM search to determine $\epsilon_\perp$ from 1st sample ($\epsilon_\parallel = 165$)</td>
<td>75</td>
</tr>
<tr>
<td>5.2</td>
<td>FEM search to determine $\epsilon_\parallel$ from 2nd sample ($\epsilon_\perp = 89$)</td>
<td>75</td>
</tr>
</tbody>
</table>
### LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>A semi–infinite 1D periodic medium</td>
<td>9</td>
</tr>
<tr>
<td>2.2</td>
<td>Bloch diagram of a periodic layered medium with unit cells consisting of (a) a single anisotropic layer (b) two anisotropic layers (c) two anisotropic and one magnetic layers</td>
<td>11</td>
</tr>
<tr>
<td>2.3</td>
<td>(a) Finite MPC (b) Band diagram of MPC, $\epsilon_a = 125$, $\delta = 40$, $\phi_{A1} = 0$, $\phi_{A2} = -\pi/12$, $L_A = 0.5695\text{mm}$ for the A layers, $\epsilon_f = 14.5$, $\mu = 1.6701$, $\beta = 0.4497$, $L_F = 0.3795L_A$ for the F layers</td>
<td>12</td>
</tr>
<tr>
<td>2.4</td>
<td>Stack of multiple cells grouped to overcome matrix instabilities in slabs with many unit cells</td>
<td>14</td>
</tr>
<tr>
<td>2.5</td>
<td>Band diagram for different ferrite layer thicknesses</td>
<td>18</td>
</tr>
<tr>
<td>2.6</td>
<td>Gaussian pulse coupling from free space to MPC and propagation within MPC. Note that free space extent is $4 \times 10^5\lambda_0$ as opposed to $480\lambda_0$ thick MPC.</td>
<td>19</td>
</tr>
<tr>
<td>2.7</td>
<td>(a)–(c) Snapshots of $E_x$ at $t = 13.936\mu s$, $t = 19.059\mu s$, and $t = 25.51\mu s$ in case $F = 0.3795$, $D = 7500\text{m}$, $N = 12000$; (d) $E_x$ at $t = 172.52\mu s$ in case $F = 0.3782$, $D = 48000\text{m}$, $N = 2500$</td>
<td>20</td>
</tr>
<tr>
<td>2.8</td>
<td>Transmittance of the MPC for different F thicknesses</td>
<td>22</td>
</tr>
<tr>
<td>2.9</td>
<td>(a) Instantaneous power density $\mathbf{z} \cdot (\mathbf{E} \times \mathbf{H})$ snapshot in the MPC (b) Average power in the MPC for $F = 0.3795$ (c) Average power of the incident field in time domain observed at the leftmost boundary of the MPC.</td>
<td>23</td>
</tr>
</tbody>
</table>
2.10 Envelope of $E_x$ in the MPC at the carrier frequency (a) $F = 0.385$ (b) $F = 0.3795$ (c) $F = 0.3782$ ................................. 25

3.1 Typical band edges within layered anisotropic media (bottom K-$\omega$ branch is not shown for simplicity); (a) MPC exhibiting a K-$\omega$ curve with an SIP; (b) DBE exhibiting a flatter K-$\omega$ curve as opposed to that of RBE. ........................................ 27

3.2 (a) MPC unit cell structure; (b) Band diagram of the MPC as a function of incidence angle; (c) Transmittance of the semi-infinite and finite ($N = 107$) MPC slab; (d) $|E_x|$ inside the MPC for normal incidence. ........................................ 29

3.3 Problem setup of the small dipole embedded in 1D layered media .......................... 31

3.4 (a) Input impedance variation as the length of the dipole inside the MPC is varied (b) Dipole received power: comparison between the dipole in the MPC and that in the simple medium (c) Receiving patterns of dipoles within the MPC for varying frequencies .......................... 36

3.5 (a) Dipole received power as the number of layers ($N$) is changed (b) Material loss effect on the dipole received power as $\tan \delta$ is increased from $10^{-5}$ to $10^{-4}$ (c) Normalized received power for different array configurations ........................................ 39

3.6 (a) DBE crystal unit cell configuration (b) Band diagram with $\phi_{A1} = 0$ and $\phi_{A2} = \pi/4$. ........................................ 41

3.7 (a) Transmittance of the finite crystal for $x$-polarization, $N = 20$, $\phi_{A1} = 0$, $\phi_{A2} = \pi/4$; $|E_x|$ and $|E_y|$ in the crystal for (b) $x$-polarized (c) $y$-polarized incident. ........................................ 42

3.8 (a) Transmittance of the finite crystal for $x$-polarization, $N = 20$, $\phi_{A1} = -3\pi/10$, $\phi_{A2} = -\pi/20$; $|E_x|$ and $|E_y|$ in the crystal for (b) $x$-polarized (c) $y$-polarized incidence. ........................................ 43

3.9 Dipole received power in the DBE crystal relative to that in the simple medium (a) $\phi_{A1} = 0$, $\phi_{A2} = \pi/4$ (b) $\phi_{A1} = -3\pi/10$, $\phi_{A2} = -\pi/20$ .... 45
3.10 (a) Dipole received power in the DBE crystal relative to that in the simple medium, $\phi_{A1} = -3\pi/10$, $\phi_{A2} = -\pi/20$. (b) Received power by 11 element endfire array in DBE crystal, $\phi_{A1} = -3\pi/10$, $\phi_{A2} = -\pi/20$.

4.1 Two region problem for constructing the PMCHWT–SIE formulation

4.2 Biquadratic quadrilateral surface element defined by 9 points

4.3 Current distributions on surface of a uniaxial dielectric cube. The permittivities are $\epsilon_\perp = 3$, $\epsilon_\parallel = 5$ with $\hat{c}$ along $\theta = 45^0$, $\phi = 45^0$. The cube is illuminated with an $x$–polarized incident electric field from $\theta = 0^0$ and has $1\lambda_0$ edge length.

4.4 Bi–static scattered fields in the $x$–$z$ plane when the cube shown in Fig. 4.3 is illuminated with an $x$–polarized field impinging along (a) $\theta = 0^0$; (b) $\theta = 45^0$, $\phi = 90^0$.

4.5 Current distributions on the surface of a uniaxial dielectric sphere with radius $1\lambda_0$. The permittivities are $\epsilon_\perp = 5$, $\epsilon_\parallel = 9$ with $\hat{c}$ along $\theta = 45^0$, $\phi = 90^0$. The sphere is illuminated with an $x$–polarized incident field impinging from $\theta = 0^0$. Due to the $\hat{c}$ orientation, the currents are symmetric with respect to the $y$–$z$ plane.

4.6 Bi–static scattered fields in the (a) $x$–$z$ and (b) $y$–$z$ plane when the sphere in Fig. 4.5 is illuminated with an $x$–polarized field impinging along $\theta = 0^0$.

4.7 Radiation pattern of a dipole embedded in a uniaxial slab as the distinguished axis is varied in the $y$–$z$ plane. Slab permittivities are $\epsilon_\perp = 45.8$, $\epsilon_\parallel = 17.2$.

4.8 (a) Diagram of the proposed DBE antenna: A small dipole is embedded at center of the 3$^{rd}$ free space layer of a 6-layer crystal (b) Directivity patterns simulated using b) PMCHWT-SIE at 10.1 GHz c) FEM at 10.2 GHz.
5.2 Electric field components of the resonant mode concentrated within the sample having its \( \parallel \) axis along \( y \): (a) \( |E_x| \), (b) \( |E_y| \), (c) \( |E_z| \). Note that \( |E_y| \) (\( \parallel \) axis direction) is essentially zero within the sample. Hence the resonance frequency mainly depends on \( \epsilon_{\perp} \). As a result, the resonance frequency varied less than 1% for various \( \epsilon_{\parallel} \) values.

5.3 (a) G factor dependence on cavity size. (b) Manufactured rectangular cavity showing the rutile sample.

5.4 (a) 15-layered barium titanate–alumina stack (white layers are \( \text{Al}_2\text{O}_3 \) and dark layers are \( \text{BaTiO}_3 \)) with the \( \parallel \) axis oriented along \( x \). (b) 8-layered barium titanate–alumina stack with \( \parallel \) axis along \( y \).

5.5 Comparison of a resonant mode field distribution in sample loaded cavity. Left: Full wave model at 7.68GHz; Right: Equivalent model at 7.54GHz; The sample shown in Fig. 5.4(a) is placed inside the cavity as in Fig. 5.3(b).

5.6 DBE assembly composed of three repeating units: (left) top view and (right) side view. The transparent spacers are 1.25mm high quartz tubes.

5.7 Measured and simulated gain (dB) patterns of the DBE prototype: (left) \( xz \) (right) \( yz \) plane.

6.1 (a) Simple printed loop antenna. (b) Dispersion diagram of a unit cell forming the rectangular printed loop antenna; resonances of this circularly periodic structure (2 unit cells) are marked by dots. (c) Bending \( K-\omega \) diagram to shift resonance to lower frequencies. (d) Magnified view of the dispersion diagram around the band edge.

6.2 (a) Volumetric DBE crystal composed of misaligned anisotropic layers. (b) Concept of emulating the DBE crystal on a microstrip substrate using coupled and uncoupled transmission line pairs; uncoupled lines with different lengths emulate the anisotropic \( \epsilon \) tensor (phase shift difference between two lines); likewise, coupled lines (even mode odd mode impedances) emulate rotation of the anisotropic \( \epsilon \) tensor.

6.3 DBE unit cell on a 100mil thick Duroid substrate and its corresponding band diagram.
6.4 (a) MS-DBE layout on a $2'' \times 2''$, 100mil thick Duroid substrate having $\epsilon_r = 2.2$, tan $\delta = 9 \times 10^{-4}$ (b) Tangential electric field amplitude on the top surface of $2'' \times 2''$ substrate (c) $|S_{11}| < -10$dB bandwidth of the MS-DBE antenna (d) Radiation pattern of the MS-DBE antenna at 2.59GHz.

6.5 Substrate thickness effects on the MS-DBE antenna performance. Thicker substrates enhance the bandwidth by narrowing the band gap at the K-$\omega$ diagram.

6.6 MS-DBE antenna array using the element in Fig. 6.4. The array size is $5.5'' \times 5.5''$ and printed on a 250mil thick Duroid substrate. Element spacing is 300mil and 90° rotation of adjacent elements achieves dual polarization. The array has an electrical size of $1.2\lambda_0 \times 1.2\lambda_0 \times \lambda_0/18$ at 2.55GHz with 2% bandwidth and 11.7dB (realized) gain.

6.7 (a) MS-DBE layout on a $2'' \times 2''$, 500mil thick substrate having $\epsilon_r = 9.6$, tan $\delta = 3 \times 10^{-4}$; the footprint size is $\lambda_0/9.6 \times \lambda_0/9.6 \times \lambda_0/16$ at resonant frequency of 1.445GHz (b) Tangential electric field magnitude on the top surface of $2'' \times 2''$ substrate (c) $|S_{11}| < -10$dB bandwidth of the MS-DBE antenna (d) Radiation pattern of the MS-DBE antenna at 1.445GHz. Radiation efficiency is $> 95%$.

6.8 Prototype of the MS-DBE antenna shown in Fig. 6.7.

6.9 Measured gain and bandwidth of the MS-DBE antenna in Fig. 6.8.

6.10 Comparison of gain $\times$ bandwidth products of the MS-DBE antenna in Fig. 6.8.

7.1 (a) Partially coupled transmission lines emulating DBE dispersion on a uniform dielectric substrate; (b) Modified version of the partially coupled lines emulating DBE. Uncoupled lines represent a diagonal anisotropy, whereas coupled section creates a general form of (non-diagonal, misaligned) of anisotropy.

7.2 (a) Lumped circuit model of partially coupled lines in Fig. 7.1(b); (b) Different band edges obtained simply by changing the amount of capacitive coupling $C_M$ in case uncoupled section is capacitively loaded ($L_1 = C_M = L_3 = 1\text{nH, } C_1 = 10\text{pF, } C_2 = C_3 = 1\text{pF, and } L_M = 0$).
7.3 (a) Transmission line implementation of lumped DBE circuit on 100mil thick Duroid substrate; (b) Dispersion diagram of the printed DBE cell on the left. Equivalent circuit model analysis agrees well with the full wave simulation. ................................. 103

7.4 (a) Lumped element model of DBE antenna composed of circularly cascaded 2 unit cells; (b) Lumped capacitor loaded dual transmission lines implementing DBE unit cell on 125mil thick Duroid substrate and their corresponding dispersion diagram. $K = \pi$ resonances are taken to lower frequencies by switching to a DBE $K-\omega$ curve. ................. 104

7.5 (a) DBE antenna layout on a $2'' \times 2''$, 125mil thick Duroid substrate; (b) $|S_{11}| < -10\text{dB}$ resonances and bandwidths computed for different coupling capacitors. .......................................................... 105

7.6 (a) Left: Fabricated DBE antenna on $2'' \times 2''$, 125mil thick Duroid substrate with $C_M = 0.5\text{pF}$, Right: Measured gain of the DBE antenna; (b) $|S_{11}| < -10\text{dB}$ resonances and bandwidths measured for different coupling capacitors. .......................................................... 106

7.7 (a) Equivalent circuit model of a partially coupled triple transmission lines; mutual coupling is achieved via the capacitors $C_{M12}$, $C_{M23}$, and $C_{M13}$; (b) 6th order $K-\omega$ curve exhibiting symmetric stationary inflection points; $L_{ij} = 1\text{nH}$, $C_{11} = 10\text{pF}$, $C_{12} = 5\text{pF}$, $C_{13} = 1\text{pF}$, $C_{2j} = 1\text{pF}$, $C_{M12} = C_{M23} = 2\text{pF}$ (i=1, 2; j = 1, 2, 3). (c) Coupling between first and third lines ($C_{M13}$) controls the $K-\omega$ slope around the SIP frequency.108

D.1 General lumped element circuit model for a conventional transmission line. $Z_1$ and $Z_2$ are the impedances of series inductor $L_R$ and shunt capacitor $C_R$, respectively. ......................................................... 124

D.2 Equivalent lumped circuit model for the uncoupled line sections of the printed DBE circuit. Different loadings on separate branches emulate the length difference of the uncoupled lines. ........................................... 125

D.3 Equivalent lumped circuit model for the coupled line sections of the printed DBE circuit. Coupling inductors $L_M$ and capacitor $C_M$ emulates the coupling mechanisms in the closely spaced printed lines. Both branches have equal series inductor $L_3$ and shunt capacitor $C_3$ values implying that the circuit represents a symmetric line pair (i.e. each line having the same length and width) ........................................... 126
D.4 Circuit model in Fig. D.3 when output ports are open and even mode is excited. \( L_M \) are doubled due to the coherent mutual inductance. \( C_M \) is removed since no current flows across it. .......................... 127

D.5 Circuit model in Fig. D.3 when output ports are open and odd mode is excited. \( L_M \) vanishes due to negative mutual inductance. \( C_M \) is doubled due to virtual ground at the middle. .......................... 127

D.6 Circuit model in Fig. D.3 when output ports are shorted and even mode is excited. \( L_M \) are doubled due to the coherent mutual inductance. \( C_M \) is shorted. .......................... 129

D.7 Circuit model in Fig. D.3 when output ports are shorted and odd mode is excited. \( L_M \) vanishes due to negative mutual inductance. \( C_M \) is shorted. .......................... 129

D.8 Circuit model of the partially coupled triple transmission lines. The circuit is formed by cascading three uncoupled lines with three capacitively coupled lines. The coupled line section is composed of uncoupled and capacitive coupling blocks. .......................... 132

D.9 Coupling capacitor network in Fig. D.8 when output ports are (a) open (b) short circuited. .......................... 133
CHAPTER 1

INTRODUCTION

Growing demand for small, light-weight and multifunctional devices puts strong requirements on all electronic components used in mobile applications. Although semiconductor technology has been significantly improving to answer the miniaturization needs, antennas of such small and compact mobile systems continue to be large and bulky. Approaching to the fundamental gain × bandwidth product limit [1, 2] defining the optimal antenna performance is therefore a contemporary goal for radio frequency (RF) engineers. In general, designs carried out using the traditional RF materials and methodologies in general perform poorly as compared to the fundamental limits. It is therefore well recognized that extraordinary propagation properties offered by material modifications or, in other words, engineered metamaterials is the new frontier for achieving the optimal performance antennas and RF devices.

Material mixtures offer unique and extraordinary RF properties. For example, ferrites [3, 4], loaded ferrites, and ferroelectrics [5] have already been exploited in phase shifters [6], antenna miniaturization, and beam control. There is also a strong interest in new design methodologies using materials to realize new properties as in the case of periodic media or engineered metamaterials (either by introducing periodic voids or by mixing several known materials). As a result, extensive literature on circuit
and antenna applications of metamaterials has already emerged [7]. For instance, negative index of refraction associated with the left handed metamaterials allowed for sub-wavelength focusing and greater sensitivity for lens applications [8–10]. On the other hand, $0^{th}$ order resonances of printed circuit left handed metamaterials have been utilized to develop various miniature devices such as phase shifters and antennas [11–13]. Similarly, controllable dispersion properties of electromagnetic bandgap materials (or photonic crystals) [14, 15] have been utilized in novel waveguide, resonator, and filter designs [11,13]. The strong resonances provided by the defect mode photonic crystals have been shown to transform small radiators into directive antennas [16,17], while their forbidden propagation bands have been successfully exploited as high impedance ground planes to improve antenna radiation properties [18–21]. In addition to these, optimized antenna substrates that are mixtures of different dielectric materials and frequency selective surfaces have successfully been employed to improve printed antennas and array performances [22,23].

There is no doubt that material modifications offer unique and highly sought advantages for RF applications. RF engineers often seek antenna size reduction, greater focusing and sensitivity, radiation control and impedance matching. Materials that can deliver these desirable properties can therefore make a significant impact in our design capabilities. For metamaterials, one important goal is to reduce the wave velocity or even reverse it in some frequency regions to achieve an electrically small RF structure [11–13, 24–27]. In this context, recently introduced metamaterial concepts based on periodic assemblies of layered anisotropic media (and possibly magnetic materials) hold a great promise for novel devices. These assemblies, namely magnetic photonic (MPC) and degenerate band edge (DBE) crystals, display higher
order dispersion (K-\omega) diagrams and therefore offer new resonant modes and degrees of freedom to optimize performance of antennas and RF applications [28,29]. For instance, MPCs display one-way transparency (unidirectionality) and wave slow-down due to the stationary inflection points (SIP) within their K-\omega diagrams [30]. In addition, faster vanishing rate of the group velocity around the vicinity of SIP frequency (as compared to the regular band edges (RBE) of ordinary crystals where zero group velocities occur) causes a stronger resonance suitable for high gain antenna apertures. DBE crystals, which are simply obtained by removing the magnetic layers of MPCs, exhibit higher order band edges (4th vs. 2nd in RBEs) which can result in much thinner resonators suitable for conformal high gain antennas [29]. Moreover, utilizing the MPC and DBE dispersions on uniform microwave substrates (in contrast to layered media) may enable their unique properties for novel printed devices which can readily be integrated to the existing RF technology such as microwave filters, amplifiers, or printed antenna arrays.

Earlier studies of MPC and DBE crystals [28–30] concentrated on the analysis of wave propagation within infinite or semi-infinite assemblies. Moreover, they have not considered the effects of realistic materials and their losses. Thus, the understanding of MPC and DBE crystal performances within practical settings have not been established and their potential for possible RF applications were not considered. This dissertation is therefore first to investigate the fundamental concepts of realistic MPC/DBE assemblies and use them to realize novel conformal antennas. Specifically our key contributions are:

1. Demonstration of fundamental characteristics of MPC/DBE crystals in practical settings (effects of material losses and finite thickness, characteristics of
radiation from small dipoles embedded within one-dimensional (1D) crystals, etc.),

2. Development of computational tools and measurement methods to design/fabricate high gain DBE antennas,

3. Emulation of DBE properties on uniform microwave substrates via partially coupled transmission lines,

4. Introduction of a new class of printed antennas miniaturized using the DBE properties,

5. Development of lumped element circuit models for anisotropic DBE/MPC crystals to obtain convenient design guidelines and smaller antennas by combining printed lines and lumped circuit elements.

The dissertation is organized as follows:

Chapter 2 starts with fundamental properties of MPC crystals. Since these properties are directly related to the behavior of the crystals’s dispersion ($K$-$\omega$) diagram, we present a Bloch wave analysis for $K$-$\omega$ extraction [28,31]. Specifically, we demonstrate that MPCs display 3rd order $K$-$\omega$ behavior exhibiting a stationary inflection point (SIP) [28]. For the incoming RF pulses, an SIP is shown to imply one-way transparency, significant wave slow down, spatial compression and concurrent amplitude increase [30,31]. To demonstrate these unique properties, we carry out a stabilized transfer matrix formalism for 1D periodic anisotropic media consisting of multiple material layers [31,32]. In addition, we consider practical material settings to identify the effect of material losses and thicknesses on the frozen mode regime.
Having established that the MPC resonance is still observed in real low-loss materials, Chapter 3 carries out a spectral domain method of moment (SD-MoM) solution to study high gain MPC antenna applications [33, 34]. Also, this chapter presents fundamental properties of the DBE crystals (simpler modifications of MPCs without the magnetic layers) and considers their antenna applications. It is shown that DBE crystals are associated with a maximally flat band edge [29] implying higher $Q$ resonances [35] suitable for high gain antenna applications. Several numerical examples are presented to demonstrate directive radiation from small dipoles and arrays when embedded within finite thickness 1D MPC or DBE crystals.

Chapter 4 proceeds with the design of high gain three–dimensional (3D) (finite thickness and cross section) DBE assemblies. Specifically, we implement [36] an MoM solution of surface integral equation (SIE) by using the closed form dyadic Green’s functions (GF) of uniaxial dielectrics [37,38]. To our knowledge, this is the first time that uniaxial GFs are used within SIE context and significant advantages in terms of MoM matrix filling time is obtained in contrast to the numerically costly spectral domain uniaxial GFs [39–42]. Using this approach we numerically demonstrate that small dipole antennas can indeed radiate directive beams (similar to the 1D case) once the 3D crystal lateral size is carefully designed [43].

Chapter 5 considers possible choices of real anisotropic materials for 3D DBE fabrication. Since DBE crystals require low loss anisotropic materials for high radiation efficiencies, a highly resonant rectangular cavity based measurement setup is developed to characterize dielectric constants and loss tangents of uniform or engineered uniaxial materials. Although several highly resonant cavity based measurement setups providing high accuracy had already been reported both for isotropic [44,45],
and uniaxial [46] dielectric materials, our approach [47] extends the one in [46] to rectangular prism-shaped material samples and incorporates finite element method (FEM) [48] to handle engineered uniaxial crystals [49]. As an example, we characterize low loss uniaxial rutile crystal and verify that it is indeed a good candidate for a DBE realization. However due to its high cost, an alternative uniaxial material, made up from fine stacks of barium titanate (BaTiO$_3$) and alumina (Al$_2$O$_3$), is proposed. We identify a low-loss adhesive type (in terms of material losses) for the titanate-alumina layers of the engineered uniaxial stack [50]. Using these materials, we perform an experimental verification of a 3D DBE crystal and show that a small slot radiator is converted into a directive antenna with almost perfect aperture efficiency.

In Chapter 6, we propose the novel concept of partially coupled dual transmission lines to emulate propagation properties of DBE crystals on otherwise uniform microwave substrates [51–53]. The dual printed line emulation enables straightforward and low cost realization of novel RF devices that can harness the DBE properties. As an important application area, we consider DBE modes to design a new class of printed miniature antennas. To do so, we use the maximally flat (4$^{th}$ order) DBE $K$-$\omega$ relation to lower the antenna resonant frequency. We first introduce a design methodology to form layouts of the DBE antennas. Subsequently, we fabricate a miniature DBE antenna [53] and compare its performance to the fundamental gain $\times$ bandwidth product limit [1,2] and several other metamaterial based antennas [24–26]. Specifically, we demonstrate that the $\lambda_0/9 \times \lambda_0/9$ footprint DBE antenna performs almost at the optimum limit with 3% bandwidth and 4.5dB realized gain.
Since printed DBE designs require many layout iterations and does not allow for a detailed understanding of different coupling mechanisms (i.e. capacitive, inductive, or both), the last chapter introduces a lumped DBE circuit model [54]. Using the lumped model, we derive design guidelines to enhance the amount of resonator/antenna miniaturization afforded by the maximally flat DBE K-ω curve. Following the guidelines, we carry out a DBE antenna design example via appropriate combinations of printed lines and lumped elements. In addition, we introduce a 6-port lumped element circuit model of triply coupled transmission lines to realize 6th order K-ω curves. Most importantly, we demonstrate, for the first time, that 6th order K-ω curves exhibit symmetric SIPs which were previously known to exist only within the presence of biased ferrites.

The dissertation concludes with a summary of major contributions, potential RF applications of DBE assemblies and provide guidelines for future research topics in this novel area of MPC and DBE crystals.
CHAPTER 2

RF PROPAGATION WITHIN
1D MAGNETIC PHOTONIC CRYSTALS

This chapter presents an analysis of magnetic photonic crystals (MPCs) constructed from periodic arrangements of available anisotropic material layers. Earlier, analytical studies of these crystals demonstrated that they exhibit the phenomena of minimal reflection at their interface, large amplitude growth of the harmonic wave within the crystal, and concurrent group velocity slow-down [28, 30]. These are associated with the so called frozen mode and occur at a specific frequency associated with a stationary inflection point (SIP) within the dispersion (K-ω) diagram. At RF frequencies, such composites also allow for the realization of electromagnetic unidirectionality (a phenomenon which permits propagation only in one of two opposite directions). The one way transparency and wave slow-down, coupled with large field amplitudes was already presented in [30] for a semi infinite slab (halfspace). Thus, the analysis in [30] did not allow for an understanding of the MPC in a practical setting. Towards this goal, here we consider the propagation of electromagnetic pulses through a finite width crystal slab and propose a realizable combination of materials consisting of available ferrite and dielectric media. The existence of significant wave
amplitude growth and slow down are verified for materials with realistic losses. In addition, we identify and characterize the bandwidth of the MPC crystals and examine its relationship to amplitude growth [31].

2.1 Characteristics of MPCs

RF propagation properties of periodic media can effectively be described using Bloch diagrams [15]. For example, let us consider a one-dimensional (1D) periodic medium oriented along the $z$ direction as shown in Fig. 2.1. Allowable propagation bands for this structure can be determined by making use of Bloch’s theorem together with the transfer matrix ($T$) of the unit cell (a periodic element of the crystal). In accordance with Bloch’s theorem (an $e^{-i\omega t}$ time convention is assumed and suppressed), electromagnetic fields propagating along the $z$ axis can be represented as a linear superposition of the Bloch eigenmodes satisfying the periodic relation

$$\begin{bmatrix}
E(z+L) \\
H(z+L)
\end{bmatrix} = \begin{bmatrix}
E(z) \\
H(z)
\end{bmatrix} e^{ikL}, \quad (2.1)$$

Figure 2.1: A semi–infinite 1D periodic medium
where \( k \) is the propagation constant and \( L \) is the length of the unit cell. Since the transfer matrix of a unit cell \( \mathbf{T}(z + L, z) \) relates the field at \( z \) to the field at \( z + L \), the above equation can be recast into

\[
\left[ \mathbf{T}(z + L, z) - \mathbf{I} e^{i k L} \right] \begin{bmatrix} \mathbf{E}(z) \\ \mathbf{H}(z) \end{bmatrix} = 0,
\]

where \( \mathbf{I} \) is the identity matrix. We readily identify from (2.2) that the eigenvalues of \( \mathbf{T} \) are \( e^{i k L} \). Unity magnitude for \( k \) real corresponds to propagating waves, whereas non–unity eigenvalues are associated with evanescent waves. We note that (2.2) leads to the same fields for \( k \) and \( k + 2\pi n/L \) (\( n \) is an integer). Thus, it is convenient to restrict \( k \) within the range \( -\pi/L \leq k \leq \pi/L \) without loss of information. This region is referred to as the first Brillouin zone, and can be scaled within the range of \( -\pi \) to \( \pi \) by introducing the dimensionless wavevector \( \mathbf{K} = kL \). Every \( \mathbf{K} \) value outside the first Brillouin zone can be plotted within the range of \( -\pi \) to \( \pi \) by shifting it in multiples of \( 2\pi \). \( \mathbf{K} \) is referred to as the Bloch wavenumber, and the corresponding eigenvector defines the polarization of the field mode for that wavenumber.

Fig. 2.2(a) is an example of the reduced zone representation of a Bloch diagram for a typical anisotropic continuous medium in 0–20 GHz frequency range. This representation is also referred to as the dispersion (or \( \mathbf{K}\omega \)) relation for the subject medium. In this medium, four different Bloch modes may exist, two of them propagate along \( +z \) direction and the other two along \( -z \). Each of the modes form a branch in the dispersion diagram and their corresponding group velocities may be found by taking the derivatives of the dispersion curve with respect to the wavenumber. For reciprocal materials (symmetric \( \mathbf{K}\omega \) diagram), the \( +z \) and \( -z \) propagating modes have the same group velocities. Fig. 2.2(b) shows the (reduced) Brillouin zone dispersion diagram for a reciprocal \emph{periodic} medium where two anisotropic layers form
Figure 2.2: Bloch diagram of a periodic layered medium with unit cells consisting of (a) a single anisotropic layer (b) two anisotropic layers (c) two anisotropic and one magnetic layers

the unit cell. As seen, for this periodic crystal there are no propagating waves within a certain frequency interval, referred to as the bandgap region. A more interesting $K-\omega$ relation is observed for the MPC medium shown in Fig. 2.2(c). Of importance is that within the MPC, the group velocities of the $+z$ and $-z$ propagating waves are different from each other at the same frequency. Clearly, this kind of behavior implies nonreciprocal responses to incident radiation and is essential in realizing the unidirectionality property of such crystals.

To examine the propagation characteristics of a wave within finite length MPCs made up of $N$ unit cells as shown in Fig. 2.3(a), in the following, we consider the simplest possible MPC structure that can exhibit an asymmetric dispersion relation.
The unit cell is composed of two identical misaligned anisotropic dielectric layers (the A layers) and a ferromagnetic layer (the F layer). The magnetization direction of the F layer is also shown in Fig. 2.3(a). The constitutive tensors for the A layers are of the form

$$\bar{\varepsilon}_A = \varepsilon_0 \begin{bmatrix} \epsilon_a + \delta \cos 2\phi_A & \delta \sin 2\phi_A & 0 \\ \delta \sin 2\phi_A & \epsilon_a - \delta \cos 2\phi_A & 0 \\ 0 & 0 & \epsilon_{az} \end{bmatrix}, \quad \overline{\mu}_A = \mu_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (2.3)$$

and likewise the F layers have the material properties as

$$\bar{\varepsilon}_F = \varepsilon_0 \begin{bmatrix} \epsilon_f & 0 & 0 \\ 0 & \epsilon_f & 0 \\ 0 & 0 & \epsilon_{fz} \end{bmatrix}, \quad \overline{\mu}_F = \mu_0 \begin{bmatrix} \mu & i\beta & 0 \\ -i\beta & \mu & 0 \\ 0 & 0 & \mu_{fz} \end{bmatrix}. \quad (2.4)$$

We note that the F layers may also consist of gyroelectric materials, but in the microwave range our interest is in gyromagnetic media as given above. Fig. 2.3(b) shows an example band diagram for the infinite MPC medium. Of particular interest is the inflection region, and with a proper unit cell design the inflection region can be tuned to have a “stationary” inflection point (SIP) associated with the conditions $\omega'(K) = \omega''(K) = 0$, and $\omega'''(K) \neq 0$. This region of the band diagram is referred to as the frozen mode regime since (among other characteristics) the group velocity approaches to zero at frequencies near the SIP. Hence, the RF pulse slows down and
its amplitude grows significantly (see Fig. 2.1 for illustration). Below, we consider the analysis of a finite slab composed of the MPC unit cell with particular emphasis on wave propagation near the frozen mode regime.

2.2 Analysis of 1D Finite Thickness MPCs

Electromagnetic wave propagation in multi–layered anisotropic media can be efficiently analyzed using the transfer matrix formalism [32]. In this context, the field values at the layer boundaries can be calculated through successive multiplications of the specific layer transfer matrices. To carry out the analysis of a layered medium such as the one in Fig. 2.3(a), we start by expressing the fields as a sum of four plane wave modes: $v_1 e^{ik_1 z}$, $v_2 e^{-ik_1 z}$, $v_3 e^{ik_2 z}$, and $v_4 e^{-ik_2 z}$, where $k_1$ and $k_2$ are the supported wavenumbers and $v_j$ denotes the $j^{th}$ electromagnetic eigenmode in free space (see [28,55] for the derivation of the plane wave modes and transfer matrices). Specifically, we represent the incident, reflected, and transmitted waves at the boundaries of the MPC as

$\begin{bmatrix} E_i \\ H_i \end{bmatrix} = a_i^{(1)} v_1 + a_i^{(2)} v_3,$

$\begin{bmatrix} E_r \\ H_r \end{bmatrix} = a_r^{(1)} v_2 + a_r^{(2)} v_4,$

$\begin{bmatrix} E_t \\ H_t \end{bmatrix} = a_t^{(1)} v_1 + a_t^{(2)} v_3,$

(2.5)

in which $a_i^{(1)}$, $a_i^{(2)}$, $a_r^{(1)}$, $a_r^{(2)}$, $a_t^{(1)}$, and $a_t^{(2)}$ are the unknown coefficients. We proceed to determine these coefficients by introducing the unit cell transfer matrix $\overline{T}$ as

$\begin{bmatrix} a_r^{(1)} \\ a_r^{(2)} \\ a_t^{(1)} \\ a_t^{(2)} \end{bmatrix}^T = \left[ (-\overline{T}^N v_2) \ (-\overline{T}^N v_4) \ v_1 \ v_3 \right]^{-1} \cdot \left[ a_i^{(1)} \overline{T}^N v_1 + a_i^{(2)} \overline{T}^N v_3 \right],

(2.6)

in which $N$ refers to the total number of unit cells. To obtain the intermediate field distribution along the crystal, we can successively multiply the field at the beginning of the structure with the unit cell transfer matrix $\overline{T}$. Appendix A presents the derivation of the transfer matrix and eigenmodes of a 1D homogenous layer with
Figure 2.4: Stack of multiple cells grouped to overcome matrix instabilities in slabs with many unit cells

anisotropic material properties. $\mathbf{T}$ is then obtained by successively multiplying the transfer matrices of the layers within the unit cell structure.

Although the above procedure is simple, numerical instabilities appear when the MPC slab consists of many periodic layers. This is especially true when the excitation is close to the bandgap frequencies. We can overcome this difficulty by dividing the crystal into smaller multicell stacks and by subsequently applying the transfer matrix analysis to each of these stacks separately. Recombination of all equations into a single matrix system results in a numerically stable and efficient scheme for large $N$. As an example, let us consider the three section region shown in Fig. 2.4. For this specific case, we express the intermediate field values in terms of the plane wave modes within the F layer assuming all subdivisions end with the last layer, and begin with the first layer of the unit cell. The F layer modes can be represented as a combination of the plane wave modes $e^{i k_{f1} z}$, $e^{-i k_{f1} z}$, $e^{i k_{f2} z}$, and $e^{-i k_{f2} z}$ ($k_{f1}$ and $k_{f2}$ being the supported wavenumbers). The unknown coefficients $a_{m}^{j}$ and $b_{m}^{j}$ for the $j^{th}$ mode in
the $m^{th}$ interval can then be related to $e_{f1}$, $e_{f2}$, $e_{f3}$, and $e_{f4}$ by solving the system

$$
T_1[a_i^{(1)}v_1 + a_i^{(2)}v_3 + a_i^{(1)}v_2 + a_i^{(2)}v_4] = [a_1^{(1)}e_{f1} + a_1^{(2)}e_{f3} + b_1^{(1)}e_{f2} + b_1^{(2)}e_{f4}], \tag{2.7}
$$

$$
\overline{T}_2[a_1^{(1)}e_{f1} + a_1^{(2)}e_{f3} + b_1^{(1)}e_{f2} + b_1^{(2)}e_{f4}] = [a_2^{(1)}e_{f1} + a_2^{(2)}e_{f3} + b_2^{(1)}e_{f2} + b_2^{(2)}e_{f4}], \tag{2.8}
$$

$$
T_3[a_2^{(1)}e_{f1} + a_2^{(2)}e_{f3} + b_2^{(1)}e_{f2} + b_2^{(2)}e_{f4}] = [a_i^{(1)}v_1 + a_i^{(2)}v_3], \tag{2.9}
$$

for the positive and negative propagating modes respectively, where $v_j$’s are again used to represent the free space plane wave modes. We remark that the coefficients $a_i^{(1)}$, $a_i^{(2)}$, $a_r^{(1)}$, $a_r^{(2)}$, $a_t^{(1)}$, and $a_t^{(2)}$ are the same as those defined in (2.6). Knowledge of all unknown coefficients yields the fields at the interfaces of the multicell stacks. Within each stack, the fields can be found by successive $T$ multiplications with the field values at the incidence boundary of the stack. Thus, errors due to successive multiplications are confined within each individual multicell stack and instabilities are avoided. Once the field distributions for multiple frequencies have been determined, fields in time domain can be generated via Discrete Fourier Transforms [56].

### 2.3 Propagation in Lossless MPC

In [28] and [30], Bloch wave analysis was used to demonstrate the strong bulk asymmetry in the frequency spectrum of ideal lossless (infinite) MPCs with the existence of SIP associated with the conditions $\omega'(K) = 0$, $\omega''(K) = 0$, and $\omega'''(K) \neq 0$. At SIP, the Bloch modes of the MPC degenerate into general divergent Floquet modes which are also referred to as the frozen modes due to the vanishing group velocity. Nevertheless for practical applications, it is of interest to consider finite thickness crystal slabs. In this section, we examine propagation within the frozen mode regime for the case of a lossless finite MPC slab. Specifically, we demonstrate coupling, amplitude growth, propagation speed, and field distribution of these frozen modes when
an RF Gaussian pulse is incident (see Fig. 2.1) to an MPC formed from realistic material layers.

The A layer constitutive tensors in (2.3) represent anisotropic dielectric materials that are aligned with an angle $\phi_A$ about their primary $z$ axis. Among the naturally available materials, we will consider “rutile” (a uniaxial crystal) for the A layers. Rutile ($\text{TiO}_2$) has a dielectric constant of $\epsilon_a = 125$ and $\delta = 40$ with corresponding loss tangents of $\tan\delta_{\epsilon_a} = 9.32 \times 10^{-5}$ and $\tan\delta_{\delta} = 1.2125 \times 10^{-4}$, respectively [57]. On the other hand, the F layer tensors in (2.4) describe gyromagnetic materials with a primary axis along $z$. The imaginary off–diagonal entries describe the well known Faraday rotation effect with the ratio $\beta/\mu$. This ratio and the total length of the layer determines the amount of Faraday rotation the wave would acquire upon crossing the layer. Properly biased soft ferrites, such as yttrium iron garnet (YIG), can be used to construct the F layers.

The permeability tensors of ferrites can be well approximated by the Lorentz curves as (see [58])

$$\bar{\mu}_F = \mu_0 \begin{bmatrix} \mu & i\beta & 0 \\ -i\beta & \mu & 0 \\ 0 & 0 & \mu_{fz} \end{bmatrix} = \mu_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \mu_0 \frac{\gamma M_0}{(\omega_0^2 - \omega^2)} \begin{bmatrix} \omega_0 & i\omega & 0 \\ -i\omega & \omega_0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.10}$$

where,

$$\omega_0 = \omega_H - i\alpha \omega, \ \omega_H = \gamma H_0, \ \alpha = \frac{\gamma \Delta H}{2\omega}, \ \gamma \approx 1.4g \ \frac{\text{MHz}}{\text{Oe}}$$

with $\omega_H$ being the magnetic resonance frequency, $\gamma$ is the gyromagnetic ratio obtained from the experimental factor $g$, $\Delta H$ is the linewidth of the Lorentz curve, and $\alpha$ is the parameter associated with dissipation within the crystal. Also, $M_0$ is the usual saturation magnetization parameter of the ferrite and $H_0$ is the static magnetic bias field. It is important to note that $H_0$ must be large enough to put the ferrite into
saturation. For the $F$ layers, we will consider a narrow linewidth ferrite. More specifically, Calcium Vanadium Garnet (CVG) with $\Delta H = 1$ Oe, $g = 2.01$, $M_0 = 1950$ G will be assumed to form the $F$ layers. Also, for our analysis a narrowband pulse will be considered. Since we operate away from the ferromagnetic resonance region (to keep losses at a minimum) and the pulse is narrow band, the permeability tensor can be assumed constant over the analysis band. This allows for a constant loss tangent as well. Specifically, at 10 GHz, with $H_0 = 5.295$ kOe, we note that the $F$ layer permeability parameters to be used are $\mu = 1.6701$, $\beta = 0.4497$ with the loss tangents given by $\tan \delta_\mu = 0.0001$, $\tan \delta_\beta = 0.00034$. Also, the dielectric constant of CVG is $\epsilon_f = 14.5$ with $\tan \delta = 0.0001$.

To consider wave propagation within the MPC, we start our analysis by first assuming no losses within the crystal (losses will be considered in the subsequent section). Specifically, we set the $A$ layer material parameters to $\epsilon_a = 125$, $\delta = 40$ with $\phi_{A1} = 0$, and $\phi_{A2} = -\pi/12$. For the CVG $F$ layers the material parameters are set as above ($\epsilon_f = 14.5$, $\mu = 1.6701$, and $\beta = 0.4497$). Furthermore, the thickness of the $F$ layer is scaled relative to that of the $A$ layer. That is, a thickness of $F = 0.5$ implies that the $F$ layer is half as thick as the $A$ layer. For our examples, the $A$ layer thicknesses were set to 0.5695mm.

Fig. 2.5 provides a closer view of the band diagram for various $F$ layer thicknesses near the frozen mode frequency. As the thickness of the $F$ layer is varied, the bending of the curves change around the inflection region. For the case of $F = 0.39$, although the curve does indeed have an inflection point ($\omega''(K) = 0$), we note that it is not stationary. However, by adjusting the aspect ratio of the layers, the slope of the curve can be made to approach zero, $\omega'(K) \to 0$. Specifically, the case of $F = 0.3782$
Figure 2.5: Band diagram for different ferrite layer thicknesses leads to a curve that nearly exhibits the SIP at $K = 0.945\pi$. It is important that we operate within the minimal dispersion region (almost linear segment) of the $K$-$\omega$ curve near the frozen mode point to avoid dispersion. Hence, to a great degree, the extent of the linear segment around the frozen mode frequency (to be referred to as the frozen mode regime) defines the bandwidth of the crystal if we chose to retain the same biasing magnetic field. As an example, for $K = 0.3782$ the linear segment occurs approximately within the region $0.935\pi < K < 0.958\pi$ of the $K$-$\omega$ curve. Outside this frozen mode regime propagation within the crystal is highly dispersive.

Let us now consider the propagation of a Gaussian pulse in a slab consisting of a periodic arrangement of the crystal layers ($N = 12000$) corresponding to $F = 0.3795$ (dashed curve in Fig. 2.5). Fig. 2.6(a)–(f) demonstrates the propagation of an $x$–polarized incident Gaussian pulse into the MPC medium at the inflection region. Due to the anisotropic nature of the A and F layers, the wave inside the MPC has a general elliptical polarization and only the $x$–polarized field components are depicted in Fig. 2.6(a)–(f). We note that $t = 0$ corresponds to the instant when the incident pulse peak is $D$ meters away from the MPC boundary (for the above
Figure 2.6: Gaussian pulse coupling from free space to MPC and propagation within MPC. Note that free space extent is $4 \times 10^5 \lambda_0$ as opposed to $480\lambda_0$ thick MPC.

case $D = 7500m)$. The incident pulse has a spatial extent of $250000\lambda_0$ ($\lambda_0$ being the free space wavelength) with unit amplitude in free space and propagates towards the MPC with the speed of light. As the pulse enters the MPC, it drastically slows down (speed reduces to $c/3600$ in this case, where $c$ is the speed of light), with only a small portion of the energy being reflected. Consequently the spatial extent of the pulse inside the MPC is extremely narrowed (only about $80\lambda_0$ in the considered case), with a corresponding amplitude growth by a factor of $5.7$. Since we operate in the minimal dispersion region, the pulse propagates through the MPC medium without much distortion, and thus fields are uniformly enlarged and transmitted across the MPC slab. Table 2.1 summarizes the corresponding $D$ values as well as the spatial variances of the incident Gaussian pulses ($\sigma$) for different F layer thicknesses. The envelope of the Gaussian pulse is thus expressed as $e^{-(z+D)^2/2\sigma^2}$. We note here that, since the
minimum dispersion bandwidth of the MPC varies with the F layer thickness, the spatial extent of the incident pulse, $\sigma$, must also be adjusted accordingly. This leads to different $D$ values to ensure that at $t = 0$, $D$ is far enough so that most of the incident pulse is outside of the MPC.

Fig. 2.7(a)–(c) gives a more expanded and detailed history of the incident pulse coupling by considering the initial part ($0\lambda_0$–$4\lambda_0$) of the MPC in Fig. 2.6. Specifically, Fig. 2.7(a) shows the pulse at $t = 13.936\mu s$ whereas Fig. 2.7(b) and (c) gives the pulse shape at $t = 19.059\mu s$ and $t = 25.51\mu s$, respectively. Here, we are particularly interested in observing how quickly the pulse reaches its maximum field strength once inside the crystal. In this example, the pulse reaches its maximum amplitude within $4\lambda_0$ ($\approx 12\text{cm}$). As shown in Fig. 2.7(d), by adjusting the thickness of the ferrite layer to be $F = 0.3782$ (implying a flatter slope around the inflection point) the pulse can reach
Table 2.1: Propagation characteristics of lossless MPCs

<table>
<thead>
<tr>
<th>F</th>
<th>$D/\sigma$ (10^3m)</th>
<th>Center Freq. ± Bandwidth (GHz)</th>
<th>Min. Thickness</th>
<th>Max. Amp.</th>
<th>Pulse Speed</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.39</td>
<td>0.21/0.045</td>
<td>9.977531 ± 0.00350</td>
<td>2.25$\lambda_0$</td>
<td>1.9</td>
<td>$c/420$</td>
</tr>
<tr>
<td>0.385</td>
<td>0.9/0.105</td>
<td>9.985065 ± 0.00150</td>
<td>2.27$\lambda_0$</td>
<td>2.52</td>
<td>$c/700$</td>
</tr>
<tr>
<td>0.3795</td>
<td>7.5/1.5</td>
<td>9.993420 ± 0.00010</td>
<td>4$\lambda_0$</td>
<td>5.7</td>
<td>$c/3660$</td>
</tr>
<tr>
<td>0.3782</td>
<td>48/11.4</td>
<td>9.995523 ± 0.000012</td>
<td>10.65$\lambda_0$</td>
<td>14.3</td>
<td>$c/25000$</td>
</tr>
</tbody>
</table>

That is, as the frequency of operation and the slope of the $K$-$\omega$ curve gets closer to the SIP, the amplitude grows remarkably.

The smallest thickness of the MPC required for the maximum amplitude is an important factor for practical realizations. For this, Table 2.1 summarizes the propagation characteristics for different F thicknesses when the incident electric field is an $x$-polarized Gaussian pulse of unity magnitude. To generate the table, we chose $N = 600$ for the cases of $F = 0.39$ and $F = 0.385$. However, the number of unit cells were increased to $N = 2000$ for $F = 0.3795$ and to $N = 2500$ for $F = 0.3782$. This ensured that the entire transmitted pulse would fit within the crystal width since the bandwidth is changing for different F layers to accommodate the minimal dispersion region. As already noted, the maximum amplitude increases significantly when the thickness of the F layer is adjusted for a flatter $K$-$\omega$ curve. The slab thickness needed to achieve the maximum pulse amplitude also increases. However, we note here that decreasing the F layer thickness further will cause the $K$-$\omega$ curve to bend inward, causing the frozen mode regime to disappear.

The remarkable and unique property of the MPC can also be seen from the last two columns in Table 2.1, where the interior pulse amplitude increases by a factor of 14 and the speed is reduced to $(1/25000)^{th}$ of the original propagation velocity. Even
for the case when $F = 0.39$, the propagation speed of the pulse decreases to $(1/420)^{th}$ of that in free space. To achieve such a slow speed using uniform homogenous media, the relative permittivity $\epsilon_r$ must be as high as 176400. Even if such a high dielectric constant structure is available, reflections and mismatches at the air–dielectric interfaces would prevent EM coupling into the material. On the contrary, the most unique aspect of the MPC is that it concurrently exhibits negligible backward reflection when operating in the minimal dispersion region near the frozen mode frequency [30]. The transmittance of the MPC calculated from the Bloch analysis for the semi–infinite case is shown in Fig. 2.8 for different F layer thicknesses and for an $x$–polarized incident field. As seen, the highest transmittance occurs at the frozen mode frequency and it is more than 70% (up to 85%) for all cases.

Slow–down and amplitude growth within the MPC are consequences of multiple reflections that occur at the frozen mode regime. The snapshots of the instantaneous power density $\hat{z} \cdot (\mathbf{E} \times \mathbf{H})$ shown in Fig. 2.9(a) for the $F = 0.3795$ case clearly depicts the effect of reflections. We observe that the power density has large positive and negative values ($-z$ directed instantaneous power density) with respect to the peak
of the incident instantaneous power (+1). The pulse is basically squeezed within the MPC and its time harmonic components reflect back and forth causing a small net positive group velocity with large field amplitudes. The average power calculated by integrating the instantaneous power over a finite spatial window is a smooth Gaussian curve as shown in Fig. 2.9(b). As expected, the maximum of the average power matches the transmitted power (≈ 80% of the average incident power for this example). Further, we remark that once the spatial extent of the transmitted pulse is scaled by the pulse propagation speed (c/3660 in this case), we simply recover the incident power curve in Fig. 2.9(c).

2.4 Material Loss Effects

For practical applications, it is necessary to examine the effects of material loss on the frozen mode phenomenon. Thus, we focus on the same MPC structure discussed above with losses incorporated within the crystal layers. Table 2.2 presents the propagation properties for a realistic MPC having a loss tangent of the order \( \tan \delta \approx 10^{-4} \) (as given in Section 2.3) and for an ultra low loss MPC with \( \tan \delta \approx 10^{-5} \).
<table>
<thead>
<tr>
<th>F</th>
<th>D/σ (10^3m)</th>
<th>Freq±BW(GHz)</th>
<th>Min. thick.</th>
<th>Max. amp.</th>
<th>Min. thick.</th>
<th>Max. amp.</th>
<th>Pulse speed</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.39</td>
<td>0.21/0.045</td>
<td>9.977531 ± 0.00350</td>
<td>1.62λ₀</td>
<td>1.62</td>
<td>2λ₀</td>
<td>1.85</td>
<td>c/420</td>
</tr>
<tr>
<td>0.385</td>
<td>0.9/0.105</td>
<td>9.985065 ± 0.00150</td>
<td>1.3λ₀</td>
<td>1.95</td>
<td>2.2λ₀</td>
<td>2.4</td>
<td>c/700</td>
</tr>
<tr>
<td>0.3795</td>
<td>7.5/1.5</td>
<td>9.993420 ± 0.00010</td>
<td>1.3λ₀</td>
<td>2.5</td>
<td>3.1λ₀</td>
<td>4.36</td>
<td>c/3660</td>
</tr>
<tr>
<td>0.3782</td>
<td>48/11.4</td>
<td>9.995523±0.000012</td>
<td>1.2λ₀</td>
<td>2.6</td>
<td>3λ₀</td>
<td>5.7</td>
<td>c/25000</td>
</tr>
</tbody>
</table>

\[ \tan \delta \approx 10^{-4} \quad \tan \delta \approx 10^{-5} \]

Table 2.2: Effect of loss on propagation characteristics

As before, the MPC is illuminated with the same incident field used to generate Table 2.1. From Table 2.2, a significant amplitude increase is again observed. However, the amplitude is reduced by more than 50% even for the case of \( \tan \delta = 10^{-5} \). This higher loss is likely due to the multiple reflections within the crystal which also results in the pulse compression. It is important to note the amount of total loss is also strongly dependent on the slope of the K-\( \omega \) curve. This implies that a trade–off between loss and the maximum amplitude can be achieved within the crystal. It is also crucial to choose low loss materials for the design of MPC and adjust the SIP to achieve maximum amplitude for the given material loss tangent.

To further illustrate the loss in the MPC and its dependence on the slope at the inflection region, we proceed by considering a time harmonic field excitation. For this case, we choose a large number of unit cells (\( N = 2000 \)) to emulate a semi–infinite medium. Fig. 2.10(a)–(c) present the envelope of the electric field as a function of position within the MPC for various F layer thicknesses. We clearly observe that the envelopes attenuate faster as the K-\( \omega \) curve is made flatter by adjusting the F layer thickness. This is likely due to the higher pulse compression implying that more bounces occur within the crystal prior to reaching a steady–state condition. We note
that a non-monotonic decay as shown in Fig. 2.10(b) and (c) is associated with the field component, whereas their power density is actually monotonic.

2.5 Remarks

This chapter demonstrated that MPCs constructed from periodic arrangements of available material layers are unidirectional and have the unique property of exhibiting dramatic wave slow-down coupled with large field amplitudes. The interesting MPC phenomena are indeed present and realized when the material layers have finite loss tangents. These phenomena occur at and near a specific frequency referred to as the frozen mode (or in other words the stationary inflection point), and can be potentially exploited for improved antenna gain, matching and miniaturization.
A key property of any photonic crystal is the slow electromagnetic modes which are associated with a vanishing group velocity $\partial \omega / \partial K = 0$. Ordinary or regular band edge (RBE) photonic crystals display this slow electromagnetic mode as the operational frequency approaches the forbidden propagation bands. The slow down causes accumulation of the electromagnetic energy and therefore, the field amplitude increases much like in the case of an enclosed resonator. However, vanishing transmittance around the band edge causes slow down phenomena to be achievable only with the Fabry–Perot transmittances resulting from finiteness. As the thickness of the RBE crystals are increased, these Fabry–Perot transmittances get narrower and shift closer to the band edge frequencies leading to larger field amplitudes within narrower bandwidths.

Magnetic photonic (MPC) and degenerate band edge (DBE) crystals [28, 29, 31] alleviate some of the issues of narrow bandwidth and small field amplitude observed within thin periodic assemblies. This is done by operating away from the bandedge (MPC) or utilizing a degenerate mode (DBE) having a much flatter $K-\omega$ response.
Figure 3.1: Typical band edges within layered anisotropic media (bottom K-ω branch is not shown for simplicity); (a) MPC exhibiting a K-ω curve with an SIP; (b) DBE exhibiting a flatter K-ω curve as opposed to that of RBE.

As was demonstrated in Chapter 2, MPCs exhibit novel slow modes with their band diagrams due to the stationary inflection point (SIP) as shown in Fig. 3.1(a). At this specific point, the group velocity vanishes faster than the RBE case and the transmittance can still be as large as 1. On the other hand, as depicted in Fig. 3.1(b), DBE is relatively flatter as opposed to the RBE. Hence, a semi-infinite DBE slab is more matched at the interface experiencing a transmittance decay rate of $\Delta \omega_d^{1/4}$ whereas the transmittance around the RBE vanishes with a rate of $\Delta \omega_d^{1/2}$, where $\Delta \omega_d$ is the difference between the band edge and the frequency of operation. Also, the group velocity decreases more quickly in the DBE ($\Delta \omega_d^{3/4}$) as compared with the rate in a RBE crystal ($\Delta \omega_d^{1/2}$). The consequence of this behavior is a higher amplitude increase in the vicinity of the DBE and SIP modes. Equally important is that the DBE and MPC crystals can be thinner than those based on the RBE to achieve the same amplitude increase.

In this chapter, we consider the transmission characteristics of 1D finite thickness MPC and DBE crystals using the transfer matrix method described in the previous chapter. Specifically, we study the performance of printed antennas embedded within 1D MPC and DBE structures operating at Fabry–Perot transmittances closest to
their higher order K-\(\omega\) resonances. To do so, we employ the spectral domain (SD) Green's function (GF) of layered media within the context of method of moments (MoM) [34, 59]. It is shown that embedded dipoles within the MPC and DBE attain significant gain enhancement with narrow beamwidths [33, 60, 61].

3.1 Transmission Characteristics of Finite Thickness MPCs

Let us consider the finite length MPC assembled from \(N\) unit cells, each consisting of two misaligned anisotropic (A) layers and a biased ferrite (F) layer as depicted in Fig. 3.2(a). The A layers are realized with rutile (\(\text{TiO}_2\)) and the F layers with calcium vanadium garnet (CVG) giving the K-\(\omega\) diagram in Fig. 3.2(b). The key characteristic of this diagram is the presence of the stationary inflection point (SIP). As was discussed in Chapter 2, the vanishing group velocity and wave slow down at SIP give rise to amplitude increase much like the case of an enclosed resonator. Since the SIP location in the K-\(\omega\) diagram is sensitive to the direction of incidence (see Fig. 3.2(c)), the velocity slow down and amplitude increase (within the MPC) is also incidence sensitive.

Typical RBE crystals also exhibit amplitude increase and wave slow down at frequencies near the band edge. Since transmittance of the incident wave into the crystal vanishes at the band edge, little or no energy is externally coupled into the RBE. In contrast, MPC exhibits near unity transmittance across the air–MPC interface at the frequency where the SIP is observed. However, it should be noted that for finite thickness (rather than semi–infinite) MPC or RBE slabs, transmittance occurs only at the Fabry–Perot resonances. These resonances are depicted in Fig. 3.2(c) for an MPC slab consisting of \(N = 107\) unit cells. It is necessary for one of the Fabry–Perot
\[ F \]
\[ \begin{pmatrix} 165 & 0 & 0 \\ 0 & 85 & 0 \\ 0 & 0 & 165 \end{pmatrix} = \begin{pmatrix} 159.64 & -20 & 0 \\ -20 & 90.36 & 0 \\ 0 & 0 & 165 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \]

\[ \Phi_1 = 0, \ \Phi_2 = -\frac{\pi}{12} \]

\[ L_{A1} = L_{A2} = 0.5695\text{mm} \]
\[ L_F = 0.3795\text{L}_{A1} \]

Figure 3.2: (a) MPC unit cell structure; (b) Band diagram of the MPC as a function of incidence angle; (c) Transmittance of the semi-infinite and finite \((N = 107)\) MPC slab; (d) \( |E_x| \) inside the MPC for normal incidence.

resonances to overlay the SIP resonance (associated with the infinite slab) for a realization of the large amplitude growth within the finite MPC. Indeed this is observed in Fig. 3.2(c) around 10GHz. Also, because the SIP occurs away from a band edge, the resulting bandwidth is substantially larger than RBE structures. Moreover, in the case of RBE, it is impossible to obtain a Fabry–Perot peak at the band edge. One can, of course, increase the thickness of the RBE slab to get the Fabry–Perot resonances closer to the band edge, but this is impractical.

From the above, we can summarize that the MPC can be suitable as a host medium for antennas that take advantage of the amplitude growth to achieve high gain. Concurrently, the high contrast dielectrics compromising the MPC imply significant antenna miniaturization. However, before discussing the antenna performance in the
presence of the MPC, it is important to point out the difference of the SIP resonance from those that were previously used for gain enhancements. Among the several structures used for gain enhancements, we remark the substrate–superstrate [62], multiple substrates [63], and the typical photonic bandgap crystal with a defect layer [16]. In all of these approaches, the gain of the embedded antennas were increased by harnessing the strong resonance that tailored to a specific location of the antenna element. In this respect, the resonance of the first two is similar to that observed at the defect layer of the PBG crystal. On the other hand, the SIP resonance occurs within the propagation band. This is seen in Fig. 3.2(c) where the merging Fabry–Perot resonances at the SIP can give a significant bandwidth beyond a single resonance. Moreover, since the SIP resonance depends on the slow mode propagation, the amplitude growth inside the crystal is more uniformly distributed (see Fig. 3.2(d)) throughout the crystal and this is important for realization of small arrays that take advantage of the entire volume. Therefore, it is essential to demonstrate and characterize the MPC not only for a single element, but also for an array of these placed inside the MPCs.

3.2 Analysis of Strip Dipoles Embedded within 1D Finite Thickness Layered Media

In this section, we focus on the radiation properties when an electrically small dipole antenna or array is placed within a finite thickness MPC or DBE slab as demonstrated in Fig. 3.3. The length to width ($L/W$) ratio of the strip dipole is large (e.g. $L/W > 10$) and it represents an equivalent of a wire dipole parallel to the layer surfaces. It can therefore assumed to be elongated along $x$ axis without loss of
Figure 3.3: Problem setup of the small dipole embedded in 1D layered media
generality. Our aim is to demonstrate that the small dipole gets translated into a
directive radiator due to the amplitude growth observed at the SIP or DBE resonance.

Radiation from strip dipoles embedded within layered media can conveniently be
analyzed through the spectral domain method of moments (SD-MoM) solution of the
electric field integral equation [40, 41, 64]. Since $L/W$ ratio is large, we assume that
dipole induced current $\mathbf{J}$ (due to the incident electric field $\mathbf{E}^i$) only varies along the
$x$ axis (i.e. $\mathbf{J} = \hat{x}J$). We further assume that the strip surface $S$ is a perfect electric
conductor (PEC). Enforcing the tangential field continuity on $S$ then leads to

\[
\hat{t} \cdot \{ \mathbf{E}^i(\mathbf{r}) + \mathbf{E}^s(\mathbf{r}) \} = 0, \quad \mathbf{r} \in S,
\]

where $\mathbf{E}^s$ is the scattered electric field at the observation point $\mathbf{r}$, and $\hat{t} = \hat{x}$ is the
tangential unit vector on $S$. Using the dyadic Green’s function $\mathbf{G}_e$ defining the electric
field generated by a delta electric current located at $\mathbf{r}'$, we can express the scattered
field as

\[
\mathbf{E}^s = \int_{S'} \mathbf{G}_e(\mathbf{r}, \mathbf{r}')\mathbf{J}(\mathbf{r}')d\mathbf{r}'.
\]

To numerically solve the equation in (3.1), we discretize the strip surface $S$ (along $x$
axis) with equal length ($h_x$) rectangles. We next enforce the current continuity condi-
tion on the shared edges of two neighboring rectangles to construct the rooftop basis
functions [65]. Assuming current flow in $+x$ direction, the rooftop basis functions $B_n$ take the form

$$B_n(x) = \hat{x}B_n(x) = \hat{x}\begin{cases} 1 - \frac{|x-P_n^x|}{h_x}, & P_n^x - h_x < x < P_n^x + h_x \\ 0, & \text{otherwise} \end{cases} \quad (3.3)$$

where $P_n^{x,y,z}$ are the midpoint coordinates of the $n^{th}$ edge defining $B_n$, and $h_y = W$ is the width of the rectangular surface elements. The surface current can now be expressed as

$$J \simeq \sum_{n=1}^{N} j_n B_n \quad (3.4)$$

where $N$ is the total number of basis functions, and $j_n$ are the unknown expansion coefficients. The MoM system [66] is generated by substituting (3.4) into (3.2), and then testing the integral equation in (3.1). After employing the typical Galerkin’s testing ($T_m = B_m$), we obtain the matrix system

$$\mathbf{Z}_e \cdot \mathbf{j} = \mathbf{b}_e. \quad (3.5)$$

The individual matrix elements of the impedance matrix and excitation vector are given by

$$\mathbf{Z}_e(m,n) = \int_S \int_{S'} T_m \cdot \mathbf{G}_e \cdot B_n d\mathbf{r}'d\mathbf{r}, \quad (3.6)$$

$$\mathbf{b}_e(m) = -\int_S T_m \cdot \mathbf{E}d\mathbf{r}, \quad (3.7)$$

in which $T_m$ are the testing functions.

Gaussian quadrature rules can be employed for the numerical evaluation of the terms in (3.6) and (3.7). Unlike homogenous isotropic media where $\mathbf{G}_e$ and $\mathbf{G}_m$ (electric and magnetic fields generated by delta electric current source) have closed
form expressions, for a 1D layered anisotropic media, $G_e$ and $G_m$ are only available in the spectral domain (i.e. plane wave expansion) as

$$
\begin{bmatrix}
G_e \\
G_m
\end{bmatrix} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ \begin{bmatrix}
G_e^0 \\
G_m^0
\end{bmatrix} + \begin{bmatrix}
\mathbf{R}_e \\
\mathbf{R}_m
\end{bmatrix} \right\} e^{i(k_z|z-z'| + k_x(x-x') + k_y(y-y'))} dk_x dk_y,
$$

$$
\begin{bmatrix}
G_e^0 \\
G_m^0
\end{bmatrix} |_{|z-z'| > 0} = \frac{-\eta_0}{8\pi^2 k_0} \begin{bmatrix}
-\eta_0 & \mp k_x & \mp k_y & \mp k_x & \mp k_y \\
\mp k_x & \pm \eta & \mp k_y & \pm k_x & \mp k_x \\
\mp k_y & \mp k_y & \pm \eta & \pm k_y & \mp k_y \\
\mp k_y & \mp k_y & \mp k_x & \pm \eta & \pm k_x \\
\pm k_x & \mp k_x & \mp k_x & \mp k_x & \pm \eta
\end{bmatrix},
\tag{3.8}
$$

where $k_0$ and $\eta_0$ are the free space wavenumber and intrinsic impedance, $k_z = \sqrt{k_0^2 - k_x^2 - k_y^2}$, $G_{xx}^0$ (where $xx = \{e, m\}$) terms are the plane waves radiated into free space, and $\mathbf{R}_{xx}$ represent the reflected fields from the layered media around the delta source (an $e^{-i\omega t}$ time convention is assumed and suppressed). Appendix B includes a detailed derivation of the spectral domain Green’s function in homogenous media and associated $G_{xx}^0$ terms. Due to the double infinite integrals and the $|z-z'| = 0$ singularity of (3.8), standard Gaussian quadrature rules are not appropriate for the numerical evaluation of the terms in (3.6).

To derive a more suitable expression for the numerical computation of the impedance matrix elements, we first substitute the basis functions in (3.3) into (3.6) and rearrange the integrals as
\[ \mathbf{Z}_e(m, n) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ \int_S B_m(x) e^{i(k_x x + k_y y)} dx \right\} \cdot GR \cdot \left\{ \int_{S'} B_n(x') e^{-i(k_x x' + k_y y')} dx' dy' \right\} dk_x dk_y, \]

where

\[ GR = \hat{x} \cdot \left\{ \left[ \begin{array}{c} \mathbb{G}_0^m \\ \mathbb{G}_0^m \end{array} \right] + \left[ \begin{array}{c} \mathbb{R}_e \\ \mathbb{R}_m \end{array} \right] \right\} e^{i k_z |P_m^z - P_n^z|} \cdot \hat{x}. \]  

(3.9)

Integration of the terms involving the testing and basis functions leads to

\[ \mathbf{Z}_e(m, n) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{B}_m(-k_x, -k_y) \cdot GR \cdot \tilde{B}_n(k_x, k_y) dk_x dk_y, \]

\[ \tilde{B}_n(k_x, k_y) = h_x h_y e^{-ik_x P_n^x} e^{-ik_y P_n^y} \text{sinc}^2(k_x h_x/2) \text{sinc}(k_y h_y/2) \]  

(3.10)

in which \( \text{sinc}(x) = \sin(x)/x \). Due to the presence of the sinc functions, the infinite integrals in above equation are now convergent provided that a proper Sommerfeld integration path satisfying the radiation condition is employed [34]. However, the numerical evaluations can be carried out with adaptive techniques, since the integrands are oscillatory for the testing and basis functions located far away (i.e. \( \gg \) wavelength) from each other.

As can be expected, another challenge of the numerical implementation is the anisotropic nature of the material layers and the existence of strong evanescent waves in the spectral integrals (i.e. \( \{k_x, k_y\} \gg k_0 \)). This results in numerical instabilities if the reflected fields around the basis functions are computed through the transfer matrix approach described in Chapter 2. To overcome these instabilities, we employ the recursive geometrical series approach [34] to determine the generalized reflection coefficients in (3.8).
3.3 Dipole Performance within the MPC

To examine the benefits of the frozen mode exhibited by the MPCs for antenna gain enhancements, a single dipole is placed within the crystal made up from the unit cells shown in Fig. 3.2(a). Since the effective dielectric constant of the MPC is about \( \varepsilon_r = 100 \), we specifically choose a dipole of length \( \lambda_0/20 \) (\( \lambda_0 = \) free space wavelength). A goal is to obtain the maximum benefit of the crystal. Therefore, we choose a rather thick MPC consisting of \( N = 107 \) layers, corresponding to a total thickness of \( 4.8\lambda_0 \). The actual \( \lambda_0/20 \times \lambda_0/200 \) strip dipole resonated at the SIP frequency of \( f_0 = 9.99342\text{GHz} \) for \( L_F = 0.3795L_A \). In the following, we compute the received power by this \( \lambda_0/20 \) dipole and compare it to the resonant \( \lambda_0/20 \) (\( \lambda_g/2 \)) dipole placed in the uniform \( \varepsilon_r = 100 \) slab. The slab thickness was also chosen to be \( 4\lambda_0 \) so that the reflectivity of the incident plane wave is minimized at the air–dielectric interface (based on the \( \lambda/4 \) transformer concept).

To obtain the power received by the \( \lambda_0/20 \) dipole within the MPC, we carried out two computations. For one of the computations, the dipole was open circuited to extract the terminal voltage (\( V_{OC} \)). In the other computation, the dipole was shorted and the terminal current (\( I_{SC} \)) was computed. The power was then evaluated by replacing the dipole with a matched Thevenin equivalent circuit (\( Z_{LOAD} = Z_{IN}^* \), \( Z_{IN} = V_{OC}/I_{SC} \)). We specifically placed the dipole at a maximum field location within the crystal corresponding to the front of the 49\(^{th}\) unit cell as depicted in Fig. 3.3. The calculations were also performed for lossless as well as lossy situations (\( \tan \delta = 10^{-4} \)).

As our first calculation, we computed the input impedance of the dipoles to ensure that a reasonable impedance is attained. We also verified that the dipoles have the
same resonance length ($\approx \lambda_0/20$) both in the MPC and the simple dielectric slab as shown in Fig. 3.4(a). At resonance ($X_m \approx 0$), we observed that $Re(Z_{MPC}) = 9.13$ and $Re(Z_{DIEL}) = 10.48$, where $Z_{MPC}$ refers to the dipole in the MPC and likewise $Z_{DIEL}$ refers to the dipole in the simple dielectric slab. These are reasonable values given the small size. Therefore we proceeded with the computation of the received power as a function of incidence.

Having calculated the received power, we found that the $\lambda_0/20$ dipole received 15dB more power when in the MPC as compared to the same dipole in a simple dielectric slab (see Fig. 3.4(b)). In addition to this significant power enhancement, the
resulting half power beamwidth of the dipole was much narrower when in MPC (14° as compared to 45° when the dipole is in the simple medium). This significant power enhancement and pattern narrowing is, of course, a direct consequence of the MPC’s sensitivity to the incidence angle. As mentioned before, the K-ω diagram shift as a function of incidence angle, leads to a more confined pattern even if the antenna is small.

The sensitivity of the inflection point to incidence angle can be advantageous for beam shaping without changing orientation or phasing of the antenna element. To demonstrate this concept, we calculated the received power of the same dipole (in MPC) as the operational frequency was increased slightly. From Fig. 3.4(c) we observe that the pattern splits into two peaks that move apart due to the occurrence of the frozen mode at different angles of incidence. Of course, once the frequency is increased beyond a certain value (where no frozen mode exists for any incidence direction), the pattern ceases to display large peaks and drops significantly down to that of a dipole in a simple medium. Another interesting property associated with the MPCs pertain to a reversed bias of the ferrite layers. Reverse–biasing changes the direction of reception by flipping the K-ω diagram about the K = 0 axis. To show the effect of reverse–biasing, we computed the dipole received power in the same MPC when the F layers was biased in the opposite direction. As expected, under reverse bias the received power was less than the case when the F layer was forward–biased (see Fig. 3.4(c)). However, due to the finiteness of the crystal, the reflections from the bottom boundary re–excite a frozen mode. Since this excitation is due to secondary reflections, the frozen mode is weaker, implying a less received power as compared to the forward MPC biasing. Nevertheless, the dipole received power still remains
larger than that when the same dipole is inside a simple medium. Specifically, the pattern is about 8dB lower and still follows the shape modified by the frozen mode.

3.4 MPC Parameter Effect on Dipole Performance

To consider the issues associated with the practical realization of the MPCs, we examine effects of crystal thickness and material loss on power reception. In decreasing the number of unit cells (N), care is given to ensure that a Fabry–Perot resonance still occurs at SIP for minimized reflection. As already noted in Chapter 2, a minimum thickness is required to observe the maximum achievable amplitude at the inflection point of an MPC. Therefore, the received power drops as the thickness of the MPC becomes smaller from N = 107 down to N = 70 and N = 30. Accordingly, the half power beamwidth increases (see Fig. 3.5(a)).

Nevertheless, it should be remarked that even in the case of N = 30 (thickness≈1.5λ₀) the MPC embedded dipole receives 6dB more power as compared to being in a simple medium. A similar drop in the received power is also observed when the material loss tangents are included in the calculations. To demonstrate the effect of loss on the received power and pattern, we started with a very small material loss tangent of tan δ = 10⁻⁵ and gradually increased it up to tan δ = 10⁻⁴. We note here that since the amplitude inside the MPC drops with the addition of loss, less cells are required to reach the maximum field amplitude inside the MPC. Therefore, the N value was also decreased as the loss factors are gradually increased. Specifically, N = 107 for tan δ = 10⁻⁵, N = 80 for tan δ = 5 × 10⁻⁵, and N = 60 for tan δ = 10⁻⁴ was used in the calculations. As depicted in Fig. 3.5(b), the received power in the normal incidence direction is reduced and the beamwidth is narrowed as the loss
Figure 3.5: (a) Dipole received power as the number of layers (N) is changed (b) Material loss effect on the dipole received power as tan δ is increased from $10^{-5}$ to $10^{-4}$ (c) Normalized received power for different array configurations

Factors are increased. More specifically, in case of tan δ = $10^{-4}$ the received power is reduced by 5.8 dB as compared to the $N = 107$ lossless MPC, but still 9.2 dB more power is received as compared to a dipole in a simple medium. Further, we observe that the half power beamwidth of the MPC embedded dipole is still much narrower ($30^0$ as compared to $50^0$ in a $2.5\lambda_0$ thick dielectric slab).

Since the MPCs allow for antenna miniaturization while maintaining a directive pattern, we also examine performance of arrays embedded inside the MPC. As an example, we consider three uniformly spaced broadside arrays: (a) a 10 element array spanning a length of $\lambda_0/2$, (b) a 10 element array of length $1.1\lambda_0$, and (c) a 20 element array of length $1.1\lambda_0$. For this analysis, the load is matched to the input impedance at $\theta = 90^0$. We specifically observe that the pattern shape is governed by the element pattern itself (Fig. 3.5(c)) and the array effect begins to appear only when the array...
length exceeds $1\lambda_0$. As expected, the total received power increases with the addition of more elements and becomes 11dB above that of a single dipole in the case of 20 element array.

3.5 Resonance and Amplitude Increase within Degenerate Band Edge Crystals

As mentioned in the introduction of this chapter, DBE crystals have relatively flatter band edges rather than the RBE crystals. In DBE crystals the group velocity vanishes faster as the operational frequency approaches to the degenerate band edge as compared to the RBE crystals. In addition to this, the decay rate of the transmittance around the DBE is smaller as compared to the RBE. Therefore, DBE crystals allow for a higher amplitude increase in the vicinity of band edge and can deliver substantial gain over regular crystals when they are operated at a Fabry–Perot resonance. It is therefore equally important to discuss antenna miniaturization and gain enhancements within the DBE crystals as was done in the MPC case.

DBE crystals have a very similar unit cell structure to the MPCs. One of the simplest structure that can develop a degenerate band edge is the unit cell configuration shown in Fig 3.6(a). However unlike MPCs, DBE crystals do not require magnetic layers and therefore their F layers consist of isotropic dielectric layers (or simply free space layer). The DBE crystal that is considered throughout this section consists of N unit cells, each unit cell including two identical dielectric layers that are misaligned with respect to each other (A layers, as was in the MPC case) and one free space layer (F layer). Fig. 3.6(a) and (b) present the DBE unit cell parameters and the corresponding band diagram, respectively.
Figure 3.6: (a) DBE crystal unit cell configuration (b) Band diagram with $\phi_{A1} = 0$ and $\phi_{A2} = \pi/4$.

Before moving into antenna performance within DBE crystals, let us consider the amplitude increase around DBE. To observe the effect of thickness and crystal orientation on amplitude increase, we investigate three different crystals. For the case of $N = 20$, Fig. 3.7(a) depicts the Fabry–Perot resonances in the 8–15GHz range when the crystal is illuminated with an $x$–polarized incident field. The closest Fabry–Perot peak to the DBE occurs at 11.196GHz, and we specifically observe that $|E_x|$ and $|E_y|$ reach to amplitudes of 2.6 and 2.9, respectively inside the crystal. When the illumination is $y$–polarized, $|E_x|$ and $|E_y|$ attains amplitudes of 2.9 and 3.3. (see Fig 3.7(b) and (c)). Depending on the application, it can be more desirable to make the crystal transparent to only one polarization. This can be achieved by rotating the whole crystal about its center axis. Fig. 3.8(a) depicts the transmittance of the same DBE crystal when it is rotated by an angle of $-3\pi/10$ about its center axis. As clearly seen, the rotated crystal is almost transparent to the $x$–polarized incident field. Accordingly the maximum amplitudes achieved inside the DBE become 4.1 and 4.3 for $|E_x|$ and $|E_y|$, respectively. In contrast, only $|E_x| = 0.5$ and $|E_y| = 0.53$ are possible for the $y$–polarized illumination (see Fig. 3.8(b) and (c)).
Figure 3.7: (a) Transmittance of the finite crystal for $x$–polarization, $N = 20$, $\phi_{A1} = 0$, $\phi_{A2} = \pi/4$; $|E_x|$ and $|E_y|$ in the crystal for (b) $x$–polarized (c) $y$–polarized incident.

DBE crystal consists of $N = 10$ unit cells and the misalignment angles are identical to the second case. As mentioned before, decreasing the total thickness shifts the Fabry–Perot resonance further away from the band edge. As a result, the closest Fabry–Perot peaks occurs at 11.036GHz and the maximum amplitudes inside the crystal becomes $|E_x| = 1.8$ and $|E_y| = 2.2$ for the $x$–polarization.

3.6 Dipole Performance in DBE Crystal

To demonstrate antenna miniaturization and directivity enhancement in DBE crystals, let us consider a very short strip dipole of length $\lambda_0/20$ ($\lambda_0$ being the free space wavelength at the operation frequency) placed within the DBE crystals considered in the previous section. The width of the dipole is again assumed to be 10
Figure 3.8: (a) Transmittance of the finite crystal for $x$–polarization, $N = 20$, $\phi_{A1} = -3\pi/10$, $\phi_{A2} = -\pi/20$; $|E_x|$ and $|E_y|$ in the crystal for (b) $x$–polarized (c) $y$–polarized incidence.

times smaller ($\lambda_0/200$) to emulate a wire dipole. We specifically choose the operating frequencies at the Fabry–Perot peaks closest to the DBE (11.196GHz for $N = 20$ and 11.036GHz for $N = 10$). The thicknesses of the crystals are only $0.93\lambda_0$ for the first two cases and $0.46\lambda_0$ for the last one. Since, it is more practical to place the dipole at the layer boundaries rather than inside the individual layers, we placed the dipoles on top of the layers where the field attains its maximum value. Specifically, the location of the dipole for the first case is the end of the 11$^{th}$ unit cell. In the second case the dipole is placed at the end of the F layer in the 10$^{th}$ unit cell. In the last case it is located at the end of the F layer of the 5$^{th}$ unit cell. Since the average dielectric constant inside the crystal is around 100, we compare the dipole performance with the same antenna embedded within an isotropic homogenous dielectric slab of $\epsilon_r = 100$. 

43
Further, we choose the thickness of the slab to be $1\lambda_0$ and $0.5\lambda_0$ to allow for a perfect matching at normal incidence.

The input impedances of the dipoles were computed via the SD-MoM implementation. Specifically, the impedances are $Z_{IN1} = 7.31 + 1.8j$, $Z_{IN2} = 7.2 + 1.4j$, and $Z_{IN3} = 7.11 + 1.8j$ for the three DBE crystal cases, respectively. In the simple slab the input impedances are $Z_{IND1} = 10.85 + 3.5j$ and $Z_{IND2} = 11.36 + 4.3j$ for $1\lambda_0$ and $0.5\lambda_0$ thicknesses, respectively. It is important to note that in spite of the large dielectrics and small antenna size; the input impedances are quite reasonable. Since the input impedances of the dipoles are very close to each other, we compare the performance by matching the dipoles perfectly ($Z_L = Z_{IN}^*$).

As expected, the amplitude increase inside the DBE crystal manifests itself in higher received power. Specifically, the dipole inside the first DBE crystal receives 8.5dB more power than the reference antenna in a simple medium when the crystal is illuminated with an $x$–polarized incident field (see Fig. 3.9(a)). We also observe that the half power beamwidth of the dipole is greatly reduced inside the first DBE crystal (from $76^\circ$ down to $10^\circ$). This effect is a direct result of the band diagram sensitivity to the incidence angle. Since the K-$\omega$ diagram and the associated Fabry–Perot resonance peak move along the frequency axis as the incidence angle deviates, the effect of amplification in the DBE crystal is only confined within a certain direction. For the second DBE crystal, the same dipole receives 11.7dB more power than that in a simple medium when the crystal is again excited by an $x$–polarized incident field. This is also expected since the second crystal is optimized for the specific polarization. In addition, when the loss factors ($\tan\delta = 10^{-4}$) of the dielectric layers are taken into account, the dipole received power is reduced by only 1.9dB in both
cases, that is the dipole displays almost the same performance in the low loss DBE crystal. When the dipole is placed inside the thin DBE crystal, the received power relative to the reference antenna drops as expected (see Fig. 3.10(a)). Nevertheless, the dipole still receives 4.8dB more power (4.5dB more when lossy) over the reference antenna. Moreover the half power beamwidth of the dipole inside the DBE is also much narrower than the reference antenna (26° to 76°).
It is important to mention the performance of possible array configurations within these crystals. Since a large propagation constant is observed in the $z$-direction due to the large effective dielectric value, endfire array configurations are attractive. Fig. 3.10(b) presents an example of an 11 element endfire array within the DBE crystal. The 11 elements of the array are $\lambda_0/20$ dipoles in length and placed in front and back of the center element located at the end of F layer of the 10th unit cell. Each elements are separated a unit cell from each other with a linear phasing of $-\pi$ in order to direct the beam along $+z$-direction (at the band edge, the phase difference across a unit cell is approximately $\pi$). The load was matched to the normal incidence impedance. As seen from Fig. 3.10(b), the endfire array allows for an additional 9.1dB increase in received power over the single element in the DBE crystal. Moreover, the side lobes of the receiving pattern are greatly suppressed as expected.

3.7 Remarks

The unique propagation characteristics of MPCs and DBE crystals are of great interest in antenna and array enhancements. Significant directivity increase is possible due to the resonance (large interior field amplitude). Accordingly, miniature elements with improved gain and directivity within these crystals are possible and this also allows for the realization of miniature high gain arrays. Furthermore, the possibility of beam steering by using the inflection region dependence on incidence angle and frequency of operation holds promise for new array designs. End fire array configurations are of particular interest for miniature high gain antenna designs. Further, although our emphasis was on the receiving properties of the small antennas, same conclusions can be made for radiating elements placed within MPCs and
DBE crystals using the generalized reaction theorem. In conclusion, MPCs and DBE crystals allow for drastic improvement in antenna radiation and receiving properties in several ways, including gain, pattern and size. Losses of the crystals do rise a significant issue. However, losses are dependent on the frequency of operation and have been found to be acceptable provided the crystal layers have a loss tangent smaller than $10^{-4}$ (i.e. $\tan \delta \leq 10^{-4}$).
CHAPTER 4

MOM-SIE SOLUTIONS FOR MODELING 3D UNIAXIAL MATERIALS USING CLOSED FORM DYADIC GREEN’S FUNCTIONS

As mentioned in Chapter 3, an important application area of DBE crystals is to enhance directivities of small antennas by harnessing the frozen modes [31,33,60,61]. However, to further miniaturize the antennas by using DBE crystals, high contrast uniaxial materials should be mixed with regular low permittivity dielectrics. Therefore, frozen mode resonances in a possible 3D finite magnetic photonic and degenerate band edge assembly will become very sensitive to size and frequency variations. As a result, designing such 3D crystals with traditional full wave numerical electromagnetic calculations (such as the finite element method (FEM) and volume integral equation (VIE) formulation)) is very challenging due to the exceedingly high computational requirements.

Surface integral equations (SIE) based on the well-known Poggio–Miller–Chang–Harrington–Wu–Tsai (PMCHWT) formulation is rather standard in modeling three-dimensional high contrast dielectric materials [67–69]. In the context of PMCHWT–SIE, interactions of the electrical and magnetic currents at the dielectric interfaces are calculated through the Green’s function of the corresponding background material,
thus eliminating a need to model highly varying fields within the medium. So far, the PMCHWT–SIE has been used to model homogeneous isotropic material regions, but has not been extended to anisotropic media as has been done for VIEs [70–72]. Although the spectral domain Green’s functions for layered anisotropic substrates have been derived and used to study radiation from stratified uniaxial media [39–42], they are not suited for a 3D method of moments (MoM) implementation of PMCHWT–SIE. This is because the spectral integrals require very long impedance matrix fill times, making them impractical.

To alleviate the large matrix fill central processing unit (CPU) time requirement, in this chapter, we consider a conformal MoM implementation of the PMCHWT–SIE employing the “closed form” spatial Green’s functions of the uniaxial materials [36]. To our knowledge, this is the first time that the closed form uniaxial dyadic Green’s functions reported in [37] and [38] have been used in a SIE-MoM context. A challenge in the implementation is the evaluation of the self impedance matrix elements and singularities due to the distinguished axis of the uniaxial media. For validation, bistatic scattering patterns of uniaxial targets are computed and compared with the well know finite element boundary integral (FE–BI) [48, 65] method solutions. A parametric study of the radiation by dipoles placed within high contrast uniaxial layers is also presented for various distinguished axis orientations. Such a scenario may be realized using piezoelectric liquid crystal materials and provides control of antenna radiation characteristics via an external DC bias. In addition, we consider radiation performance of small dipoles embedded within 3D finite DBE crystals. Specifically, we demonstrate that small dipoles embedded within 3D DBE assemblies
can indeed radiate directive beams (similar to the 1D case) provided that lateral size of the crystals are properly designed.

4.1 PMCHWT–SIE Formulation

Consider the uniaxial dielectric volume \((V_2)\) surrounded by the free space region \((V_1)\) shown in Fig. 4.1. The dielectric volume \(V_2\) is illuminated by an incident electromagnetic field \((\mathbf{E}^i, \mathbf{H}^i)\), and as usual \(\mathbf{E}\) is the electric field and \(\mathbf{H}\) refers to the magnetic field. To determine the fields \((\mathbf{E}^s, \mathbf{H}^s)\) scattered by the uniaxial body, we impose Love’s surface equivalence principle and express the total fields \((\mathbf{E}_1, \mathbf{H}_1)\) in \(V_1\) as

\[
\begin{align*}
\mathbf{E}_1 &= \mathbf{E}^i + \mathbf{E}^s = \mathbf{E}^i + \int_{S'} \mathbf{G}_e^1 \cdot \mathbf{J}^+ d\mathbf{r}' + \int_{S'} \mathbf{G}_{em}^1 \cdot \mathbf{M}^+ d\mathbf{r}' \\
\mathbf{H}_1 &= \mathbf{H}^i + \mathbf{H}^s = \mathbf{H}^i + \int_{S'} \mathbf{G}_{me}^1 \cdot \mathbf{J}^+ d\mathbf{r}' + \int_{S'} \mathbf{G}_m^1 \cdot \mathbf{M}^+ d\mathbf{r}'
\end{align*}
\]  

(4.1)

assuming that the total fields \((\mathbf{E}_2, \mathbf{H}_2)\) in \(V_2\) are zero. In (4.1), \(\mathbf{G}_e^1\) and \(\mathbf{G}_{me}^1\) are the free space dyadic Green’s functions, corresponding to the \(\mathbf{E}\) and \(\mathbf{H}\) fields radiated into \(V_1\) by a delta electric current source. Likewise, \(\mathbf{G}_{em}^1\) and \(\mathbf{G}_m^1\) are associated with the \(\mathbf{E}\) and \(\mathbf{H}\) fields due to delta magnetic current source. We note that \(\mathbf{J}^+\) and \(\mathbf{M}^+\) are the surface electric and magnetic currents, respectively, on \(S\) just inside \(V_1\). We should also note that from the boundary conditions on \(S\), \(\mathbf{J}^+\) and \(\mathbf{M}^+\) are related to the total fields as \(\mathbf{J}^+ = \mathbf{n} \times \mathbf{H}_1\) and \(\mathbf{M}^+ = -\mathbf{n} \times \mathbf{E}_1\), where \(\mathbf{n}\) is the unit outward normal on \(S\) (pointing towards \(V_1\)). Another equation system, similar to (4.1), can be constructed for \(V_2\) as

\[
\begin{align*}
\mathbf{E}_2 &= \int_{S'} \mathbf{G}_e^2 \cdot \mathbf{J}^- d\mathbf{r}' + \int_{S'} \mathbf{G}_{em}^2 \cdot \mathbf{M}^- d\mathbf{r}' \\
\mathbf{H}_2 &= \int_{S'} \mathbf{G}_{me}^2 \cdot \mathbf{J}^- d\mathbf{r}' + \int_{S'} \mathbf{G}_m^2 \cdot \mathbf{M}^- d\mathbf{r}'
\end{align*}
\]  

(4.2)
Figure 4.1: Two region problem for constructing the PMCHWT–SIE formulation again by adopting Love’s equivalence principle. As in (4.1), the \( \mathbf{G}_{\alpha\beta}^2 \) dyads stand for the uniaxial dyadic Green’s functions of \( V_2 \) (with the subscripts \( \alpha \) and \( \beta \) representing either \( \mathbf{e} \) or \( \mathbf{m} \)), whereas \( \mathbf{J}^- \) and \( \mathbf{M}^- \) are the equivalent electric and magnetic currents in \( V_2 \) tangential to \( S \). Again these currents are related to the total fields via the boundary conditions as \( \mathbf{J}^- = -\mathbf{n} \times \mathbf{H}_2 \) and \( \mathbf{M}^- = \mathbf{n} \times \mathbf{E}_2 \).

To construct the SIE, we impose tangential field continuity on \( S \) implying that \( \mathbf{J}^+ = -\mathbf{J}^- = \mathbf{J} \) and \( \mathbf{M}^+ = -\mathbf{M}^- = \mathbf{M} \), where we conveniently used \( \mathbf{J} \) and \( \mathbf{M} \) to represent the currents in \( V_1 \). Next, we proceed to combine (4.1) and (4.2) to generate a set of SIEs for solving \( \mathbf{J} \) and \( \mathbf{M} \). Specifically, in (4.1) we let the position vector \( \mathbf{r} \) approach \( S \) from the interior of \( V_1 \) and in (4.2) we let \( \mathbf{r} \) approach \( S \) from within \( V_2 \). Subsequently, imposing tangential continuity across \( S \) yields the SIEs (PMCHWT formulation)

\[
-\begin{bmatrix} E^i \\ H^i \end{bmatrix} = \int_{S'} \begin{bmatrix} \mathbf{G}_{\mathbf{e}e}^1 + \mathbf{G}_{\mathbf{e}e}^2 & \mathbf{G}_{\mathbf{e}m}^1 + \mathbf{G}_{\mathbf{e}m}^2 \\ \mathbf{G}_{\mathbf{me}}^1 + \mathbf{G}_{\mathbf{me}}^2 & \mathbf{G}_{\mathbf{me}}^1 + \mathbf{G}_{\mathbf{me}}^2 \end{bmatrix} \begin{bmatrix} \mathbf{J} \\ \mathbf{M} \end{bmatrix} d\mathbf{r}', \quad \mathbf{r} \in S. \tag{4.3}
\]

Once (4.3) is solved for the surface currents, (4.1) can be used to determine the scattered far fields using the far zone Green’s functions. Specifically, the bistatic
scattering coefficients (echo areas) are given by

\[
\sigma_\theta(dB/\lambda_0^2) = 10 \log_{10} \lambda_0^2 \left\{ \lim_{r \to \infty} 4\pi r^2 \frac{|E_\theta^s|^2}{|E|^2} \right\},
\]

\[
\sigma_\phi(dB/\lambda_0^2) = 10 \log_{10} \lambda_0^2 \left\{ \lim_{r \to \infty} 4\pi r^2 \frac{|E_\phi^s|^2}{|E|^2} \right\},
\]

(4.4)

where the subscripts \( \theta \) and \( \phi \) denote the polarization of the scattered fields in the usual spherical coordinates.

### 4.2 Uniaxial Dyadic Green’s Functions

As noted in the introduction, our focus is to solve (4.3) when \( V_2 \) is a uniaxial medium characterized by the constitutive relations

\[
\varepsilon = \varepsilon_\perp \mathbf{I} + (\varepsilon_\parallel - \varepsilon_\perp) \hat{c} \hat{c} \quad \text{and} \quad \mu = \mu_\perp \mathbf{I} + (\mu_\parallel - \mu_\perp) \hat{c} \hat{c},
\]

(4.5)

where \( \varepsilon_\parallel \) is the relative permittivity along the distinguished axis (\( \hat{c} \) or \( \parallel \) axis) and \( \varepsilon_\perp \) is along the other directions (\( \hat{a} \) or \( \perp \) axis) perpendicular to the \( \parallel \) axis. Similarly, \( \mu_\parallel \) and \( \mu_\perp \) are the relative permeability along the \( \parallel \) and \( \perp \) axis, respectively. As usual, \( \mathbf{I} \) denotes the identity dyad. The radiation from an Hertzian dipole (electric type) located in an infinite uniaxial dielectric medium were derived in [37] via spectral domain integrals. Later in [38], the results in [37] were generalized to uniaxial magnetodielectric media. For nonmagnetic (i.e., \( \mu = \mathbf{I} \)) uniaxial dielectric media, the closed form Green’s functions in [38] take the form

\[
\overline{G}_e^2 = \frac{i\omega \mu_0}{4\pi} \left\{ \left[ \nabla^2 + \varepsilon^{-1} \right] e^{ik_\perp R_e} \frac{\varepsilon_\parallel R_e}{R_e} - \left[ \frac{\varepsilon_\perp e^{ik_\perp R_e}}{\varepsilon_\parallel R_e} - \frac{e^{ik_\perp R}}{R} \right] \left[ (R \times \hat{c}) (R \times \hat{c}) \right] \right\},
\]

(4.6)
\[ \mathbf{G}_m^2 = \frac{i\omega\epsilon_0}{4\pi} \left\{ \left[ \nabla \nabla + \epsilon_\perp \mathbf{I} \right] e^{ik_0 R} - \left[ \epsilon_\parallel e^{ik_0 R} - \epsilon_\perp e^{ik_0 R^\perp} \right] \frac{(\mathbf{R} \times \hat{\mathbf{c}})(\mathbf{R} \times \hat{\mathbf{c}})}{(\mathbf{R} \times \hat{\mathbf{c}})^2} \right\} \]

\[ \mathbf{G}_me^2 = \left( \frac{e^{ik_0 R}}{4\pi R} - \frac{e^{ik_0 R^\perp}}{4\pi R} \right) (\mathbf{R} \cdot \hat{\mathbf{c}}) \left[ \frac{(\mathbf{R} \times \hat{\mathbf{c}})(\mathbf{R} \times \hat{\mathbf{c}})}{(\mathbf{R} \times \hat{\mathbf{c}})^2} \right] \]

\[ \mathbf{G}_em^2 = \left[ \mathbf{G}_me^2 \right]^T. \]

In the above, \( \mathbf{R} \) is the vector from the source location \( \mathbf{r}' \) to the observation point \( \mathbf{r} \) (see Fig. 4.1). Also, the scalar \( R = |\mathbf{R}| \), \( R_e \) is the modified distance by the uniaxial permittivity dyad, \( k_0 \) is the free space wave number, and \( k_\perp \) is the wavenumber modified by \( \epsilon_\perp \). These quantities are explicitly given by

\[ R = \mathbf{r} - \mathbf{r}', \ R = |\mathbf{R}|, \ R_e = \sqrt{\epsilon_\parallel (\mathbf{R} \cdot \mathbf{e}^{-1} \cdot \mathbf{R})}, \ k_0 = \omega\sqrt{\epsilon_0\mu_0}, \ k_\perp = k_0\sqrt{\epsilon_\perp}, \]

where \( \epsilon_0 \) and \( \mu_0 \) denote the usual free space permittivity and permeability, respectively, and \( \omega \) is the angular frequency of the time harmonic field (an \( e^{-i\omega t} \) time convention is assumed and suppressed). We remark that, as expected, for the case of \( \mathbf{\bar{c}} = \mathbf{I} \), the uniaxial dyadic Green’s functions given in (4.6)–(4.9) reduce to the free space dyadic Green’s functions

\[ \mathbf{G}_e^1 = \frac{i\omega\mu_0}{4\pi} \left[ \nabla \nabla + \mathbf{I} \right] e^{i k_0 \mathbf{R}}, \]

\[ \mathbf{G}_m^1 = \frac{i\omega\epsilon_0}{4\pi} \left[ \nabla \nabla + \mathbf{I} \right] e^{i k_0 \mathbf{R}}, \]

\[ \mathbf{G}_{me}^1 = - (1 - ik_0 \mathbf{R}) \left( \frac{e^{i k_0 \mathbf{R}}}{4\pi R^3} \right) [\mathbf{R} \times \mathbf{I}], \]

\[ \mathbf{G}_{em}^1 = \left[ \mathbf{G}_{me}^1 \right]^T. \]
4.3 MoM Implementation Using Curvilinear Geometry Modeling

To numerically solve the integral equation system in (4.3), we proceed to discretize the boundary surface enclosing the material using biquadratic quadrilateral surface elements, each defined by a set of 9 points as depicted in Fig. 4.2. Given these 9 points, a transformation between the surface element and a unit square bounded by the \( u = \pm 1, v = \pm 1 \) lines in the parametric \((u, v)\) domain can be formed as

\[
\mathbf{r}(u, v) = \sum_{m=1}^{3} \sum_{n=1}^{3} L_{mn}(u, v) \mathbf{r}_{2(m-1)+n}
\]

through the Cartesian products of the Lagrange interpolation functions \( L_{mn}(u, v) \). Each quadrilateral element supports four half rooftop basis functions expressible in curvilinear coordinates [65] as

\[
\mathbf{b}_1 = \frac{-(1-u)}{2\sqrt{G_s}} \mathbf{a}_u, \quad \mathbf{b}_2 = \frac{-(1-v)}{2\sqrt{G_s}} \mathbf{a}_v, \quad \mathbf{b}_3 = \frac{(1+u)}{2\sqrt{G_s}} \mathbf{a}_u, \quad \mathbf{b}_4 = \frac{(1+v)}{2\sqrt{G_s}} \mathbf{a}_v,
\]

where, \( \mathbf{a}_u = \partial \mathbf{r} / \partial u, \mathbf{a}_v = \partial \mathbf{r} / \partial v, \) and \( G_s = g_{uu} g_{vv} - g_{uv} g_{vu} \) with \( g_{xy} = \mathbf{a}_x \cdot \mathbf{a}_y \). The complete rooftop basis functions are constructed to enforce current continuity by combining half basis functions defined on the shared edge of two neighboring elements [73]. Assuming outward current flow from the element with the smallest number, the rooftop basis functions take the form

\[
\mathbf{B}_p = \left\{ \mathbf{b}_{\ell_1(p)}^{M_1(p)} - \mathbf{b}_{\ell_2(p)}^{M_2(p)} \right\} (-1)^s, \quad s = \begin{cases} +1, & M_1 > M_2 \\ -1, & \text{otherwise} \end{cases}
\]

where \( M_i(p) \) denotes the element number adjacent to the \( p^{th} \) shared edge and \( \ell_i(p) \) is the half basis number in the \( M_i(p)^{th} \) element. We note that these basis functions are divergence–conforming and they enforce current continuity across the shared element.
Figure 4.2: Biquadratic quadrilateral surface element defined by 9 points
edges. For future use, we further remark that integration over an element’s surface

can be carried out in the parametric space using $d\mathbf{r} = \sqrt{G_s} dudv$.

The surface currents in (4.3) can now be expressed as

$$\mathbf{J} \simeq \sum_{p=1}^{N} j_p \mathbf{B}_p \quad \text{and} \quad \mathbf{M} \simeq \sum_{p=1}^{N} m_p \mathbf{B}_p,$$

where $N$ is the total number of basis functions, and $j_p, m_p$ are the unknown expansion coefficients. As usual, the MoM system [66] is generated by substituting (4.18) into (4.3), and testing the integral equations. After employing the typical Galerkin’s

testing, we obtain the matrix system

$$\begin{bmatrix} \mathbf{b}_e \\ \eta_0 \mathbf{b}_m \end{bmatrix} = \begin{bmatrix} \mathbf{Z}_e^1 + \mathbf{Z}_e^2 & \eta_0 \left( \mathbf{Z}_{em}^1 + \mathbf{Z}_{em}^2 \right) \\ \eta_0 \left( \mathbf{Z}_{me}^1 + \mathbf{Z}_{me}^2 \right) & \eta_0^2 \left( \mathbf{Z}_m^1 + \mathbf{Z}_m^2 \right) \end{bmatrix} \cdot \begin{bmatrix} \mathbf{j} \\ \mathbf{m}' \end{bmatrix}$$

(4.19)

The individual matrix elements of the impedance matrices and excitation vectors are
given by

$$\mathbf{Z}_{xx}^i(p, q) = \int_{S} \int_{S'} \mathbf{T}_p \cdot \overline{\mathbf{G}}_{xx}^i \cdot \mathbf{B}_q d\mathbf{r}' d\mathbf{r},$$

(4.20)

$$\mathbf{b}_e(p) = -\int_{S} \mathbf{T}_p \cdot \mathbf{E}^i d\mathbf{r}, \quad \text{and} \quad \mathbf{b}_m(p) = -\int_{S} \mathbf{T}_p \cdot \mathbf{H}^i d\mathbf{r},$$

(4.21)
in which $T_p = B_p$ is the testing function. Here we have used the relation $m = m'\eta_0$ and re-scaled the matrix rows with $\eta_0$ ($\eta_0 = \sqrt{\mu_0/\epsilon_0}$) to achieve a more well conditioned system, because $G^i_m \sim G^i_e/\eta_0^2$ and $H^i \sim E^i/\eta_0$.

4.4 Singularities of the Uniaxial Dyadic Green’s Functions

Gaussian quadrature rules can be employed for a numerical evaluation of the terms in (4.20) and (4.21) when testing non-adjacent elements. However, for the self-cell terms this approach is not suitable. Specifically, when $(p = q)$, the integrals of (4.20) must be treated carefully for accurate evaluation of (4.19) (see e.g., [64,74–79]).

Evaluation of the uniaxial dyadic Green’s functions within the volumetric source region was carried out in [80–82]. However, for $(p = q)$, the corresponding surface integrals in (4.20) exhibit field singularities associated with the surface currents. To evaluate them, we first proceed to examine the singularity order of the terms in (4.6)–(4.9) and (4.11)–(4.14). From (4.6), (4.7), (4.11) and (4.12) we observe that $G^i_e$ and $G^i_m$ are $1/R$ singular after the $\nabla\nabla$ terms are distributed onto the testing and basis functions (this is standard and done by applying the divergence theorem twice). The resulting singularities are then integrable but must be carried out with proper care. For our case, we use an annihilation technique employing a simple transformation similar to the one reported in [64]. On the contrary, expressions (4.8), (4.9), (4.13) and (4.14) imply that $G^i_{me}$ and $G^i_{em}$ have $1/R^2$ type singularities. Although, the principal values of the source integrals (4.20) have been derived for isotropic dyadic Green’s functions (e.g., see [34]), to our knowledge, this is the first time a similar need has risen for the uniaxial Green’s functions. To evaluate the terms involving (4.8)
and (4.9) we implemented a numerical limiting procedure by gradually letting \( r \) approach \( r' \) along the surface normal as already explained in constructing (4.3). For the resulting double integral, the innermost integration must be carried out with utmost care. Specifically, we employed an adaptive Simpson’s rule for this purpose. The outer integrals of (4.20) were then computed using the 4-point Gaussian quadrature.

Another important issue with the integration of (4.6)–(4.9) arises when the distinguished axis of the uniaxial medium coincides with the direction of \( \mathbf{R} \). Because the unit vector \( (\mathbf{R} \times \hat{\mathbf{c}})/|\mathbf{R} \times \hat{\mathbf{c}}| \) become undefined in that case, special care is necessary for the numerical evaluation of (4.20). Specifically, as the \( \parallel \) axis coincides with \( \mathbf{R} \), \( R_e \) approaches \( R \) (i.e. \( R_e \rightarrow R \)). For this special case, (4.6)–(4.9) become

\[
\mathbf{G}_e^2 = \frac{i\omega \mu_0}{4\pi} \left\{ \left[ \nabla \nabla \mathbf{k}^2_\perp + \epsilon^{-1} \right] \frac{e^{ik_\perp R}}{R} - \left[ \frac{\epsilon_{\parallel} - \epsilon_{\perp} e^{ik_\perp R}}{2\epsilon_{\perp} R} \right] [\mathbf{I} - \hat{\mathbf{c}} \hat{\mathbf{c}}] \right\}, \tag{4.22}
\]

\[
\mathbf{G}_m^2 = \frac{i\omega \epsilon_0}{4\pi} \left\{ \left[ \nabla \nabla \mathbf{k}^2_\parallel + \epsilon_1 \mathbf{I} \right] \frac{e^{ik_\perp R}}{R} + \left[ (\epsilon_{\parallel} - \epsilon_{\perp}) \frac{e^{ik_\perp R}}{2R} \right] [\mathbf{I} - \hat{\mathbf{c}} \hat{\mathbf{c}}] \right\}, \tag{4.23}
\]

\[
\mathbf{G}_{me}^2 = -\left( \frac{\epsilon_{\parallel} + \epsilon_{\perp}}{2\epsilon_{\perp}} \right) (1 - ik_\perp R) \left( \frac{e^{ik_\perp R}}{4\pi R^2} \right) [\hat{\mathbf{c}} \times \mathbf{I}], \tag{4.24}
\]

\[
\mathbf{G}_{em}^2 = \left[ \mathbf{G}_{me}^2 \right]^T. \tag{4.25}
\]

In deriving (4.24) from (4.8), we used the identities

\[
R_e = \frac{\epsilon_{\parallel}}{\epsilon_{\perp}} (\mathbf{R} \times \hat{\mathbf{c}})^2 + (\mathbf{R} \cdot \hat{\mathbf{c}})^2,
\]

\[
R = (\mathbf{R} \times \hat{\mathbf{c}})^2 + (\mathbf{R} \cdot \hat{\mathbf{c}})^2,
\]

\[
\Rightarrow (\mathbf{R} \times \hat{\mathbf{c}})^2 = \frac{\epsilon_{\perp}}{\epsilon_{\parallel} - \epsilon_{\perp}} (R_e^2 - R^2), \quad \text{and}
\]

\[
\mathbf{I} = \hat{\mathbf{c}} \hat{\mathbf{c}} + \frac{(\mathbf{R} \times \hat{\mathbf{c}})(\mathbf{R} \times \hat{\mathbf{c}})}{(\mathbf{R} \times \hat{\mathbf{c}})^2} + \frac{[\hat{\mathbf{c}} \times (\mathbf{R} \times \hat{\mathbf{c}})][\hat{\mathbf{c}} \times (\mathbf{R} \times \hat{\mathbf{c}})]}{(\mathbf{R} \times \hat{\mathbf{c}})^2} \tag{4.26}
\]

\[
\text{together with L’Hospital’s rule. Equations (4.22)–(4.25) suggest that the singularity orders in the uniaxial Green’s functions remain the same. In the following, we present}
\]
numerical examples aimed at verifying the MoM implementation and uniaxial dyadic Green’s function singularities.

4.5 Numerical Examples

4.5.1 Uniaxial Dielectric Cube

As our first example, we consider the scattering by a uniaxial dielectric cube (shown in Fig. 4.3, and having a side length of $1\lambda_0$). The anisotropic permittivity of the cube is $\epsilon_\perp = 3$ and $\epsilon_\parallel = 5$ with the distinguished $\hat{c}$ (or $\parallel$) axis being along the $\theta = 45^\circ$, $\phi = 45^\circ$ direction in the spherical coordinate system. We will evaluate the bistatic scattering $\sigma_{\theta\phi}$ (4.4) in $x$–$z$ plane for two illuminations: 1) $x$ polarized electric field incident along $\theta = 0^\circ$, 2) $x$ polarized electric field incident along $\theta = 45^0$, $\phi = 90^0$ direction. To validate our results, we employed the FE–BI approach [65] as implemented in [83].

The far zone scattering data for the cube in Fig. 4.3 are given in Figures 4.4(a) and (b) for the two illuminations. We remark that pattern in Fig. 4.4(a) is not symmetric because the distinguished axis of the uniaxial medium is along $\theta = 45^0$, $\phi = 45^0$. Of importance is that the SIE pattern is in agreement with the reference FE–BI solution. For the second oblique illumination, we again observe a very good agreement with the FE–BI data as depicted in Fig. 4.4(b). Only a small deviation ($1\text{dB} \sim 1.5\text{dB}$) was observed below $-10\text{dB}$ of the $\sigma$ values. This is likely due to the different mesh densities of the SIE and FE–BI solutions.

4.5.2 Uniaxial Sphere

We next consider a curvilinear geometry. Specifically, we choose a dielectric sphere of radius $1\lambda_0$ with $\epsilon_\perp = 5$, $\epsilon_\parallel = 9$ and $\hat{c}$ along $\theta = 45^0$, $\phi = 90^0$. Since the distinguished
Figure 4.3: Current distributions on surface of a uniaxial dielectric cube. The permittivities are $\epsilon_\perp = 3$, $\epsilon_\parallel = 5$ with $\hat{c}$ along $\theta = 45^\circ$, $\phi = 45^\circ$. The cube is illuminated with an $x$–polarized incident electric field from $\theta = 0^\circ$ and has $1\lambda_0$ edge length.

axis is oriented in the $y$–$z$ plane, an $x$–polarized plane wave incident along $\theta = 0^\circ$ would generate currents that are symmetric about the $y$–$z$ plane (see Fig. 4.5). This is verified in the scattered field pattern depicted in Fig. 4.6(a) as it is symmetric around the $z$ axis when computed in the $x$–$z$ plane. In contrast, the $y$–$z$ plane scattered fields are asymmetric around $z$ axis as shown in Fig. 4.6(b). Again a good agreement of the bistatic echo area pattern is observed with the reference FE–BI data. However,
Figure 4.5: Current distributions on the surface of a uniaxial dielectric sphere with radius $1\lambda_0$. The permittivities are $\epsilon_\perp = 5$, $\epsilon_\parallel = 9$ with $\hat{c}$ along $\theta = 45^0$, $\phi = 90^0$. The sphere is illuminated with an $x$-polarized incident field impinging from $\theta = 0^0$. Due to the $\hat{c}$ orientation, the currents are symmetric with respect to the $y$–$z$ plane.

As expected, the computational advantage of the PMCHWT-SIE in modeling high contrast materials is rather significant (12,288 unknowns for SIE vs. 16,224 surface and 133,940 volume unknowns for the FE–BI). Due to the large memory requirement of FE–BI, a three level multi level fast multipole method (MLFMM) was employed to carry out vector matrix multiplications of the boundary integral terms. This resulted in 800MB of storage and 192 BiCGSTAB(8) iterations using 1470sec CPU time. On the other hand, SIE used 600 MBytes memory and 11 BiCGSTAB(8) taking 660 sec. in CPU time. Both solutions were run on an AMD Opteron 250 at 2.4GHz using a solution tolerance of $1 \times 10^{-3}$.

4.5.3 Dipole Radiation in High Contrast Uniaxial Slab

As the third example, we consider the radiation from a small dipole embedded within a high contrast uniaxial slab. For the dipole excitation, the incident fields in (4.19) are now those on the slab walls radiated by the impressed dipole excitation. The dipole is modeled as a thin strip using quadrilateral elements and the rooftop basis at the dipole center is forced to have a constant amplitude and phase. Also,
Figure 4.6: Bi-static scattered fields in the (a) $x$–$z$ and (b) $y$–$z$ plane when the sphere in Fig. 4.5 is illuminated with an $x$–polarized field impinging along $\theta = 0^\circ$.

The impedance matrix elements were modified by introducing the $Z_{UDe}$, $Z_{UDe}$, $Z_{USem}$, $Z_{USe}$, and $Z_{DDe}$ matrices into (4.19) along with the dipole currents. These matrices are calculated similarly to (4.20), where U represents the uniaxial slab currents and D is used for the dipole currents.

The considered uniaxial slab has $\epsilon_\perp = 45.8$, $\epsilon_\parallel = 17.2$, an aperture size of $1\lambda_0 \times 1\lambda_0$, and a height of $0.1\lambda_0$. In practice, a slab with such material parameters can be realized by a periodic arrangements of layered barium titanate ($\epsilon = 82$) and alumina ($\epsilon = 9.6$) stacks [49, 50]. The distinguished axis of the slab is assumed to be along the $y$ axis, and a $0.05\lambda_0$ length dipole is placed in the middle of the slab just $0.01\lambda_0$ above the bottom face as shown in Fig. 4.7.

Fig. 4.7 gives the radiation pattern when the distinguished axis of the uniaxial slab is varied in the $y$–$z$ plane. As seen, when the $\parallel$ (or $\hat{c}$) axis is shifted $5^\circ$ from the $y$ axis, the narrow beam exhibiting $D = 10\text{dB}$ directivity is still preserved. However, when the $\parallel$ axis is further tilted, a rotation of the main beam in the $y$–$z$ plane is observed. As a result of this abrupt change in the pattern, the directivity decreases...
Figure 4.7: Radiation pattern of a dipole embedded in a uniaxial slab as the distinguished axis is varied in the $y$–$z$ plane. Slab permittivities are $\epsilon_{\perp} = 45.8$, $\epsilon_{\parallel} = 17.2$.

However, by rotating the $\parallel$ axis more towards the $z$ axis, some of the directivity is recovered (up to 7.9 dB). That is, the dipole radiation is very sensitive to the distinguished axis orientations for high contrast uniaxial dielectrics. This is primarily due to the excitation of the structural dielectric resonances.

### 4.5.4 Dipole Radiation within 3D DBE Crystals

For the last example, we consider radiation from a small dipole placed inside a 3D DBE crystal as depicted in Fig. 4.8(a). The crystal consists of 6 unit cells, each including two misaligned uniaxial slabs and a free space layer. The uniaxial slab material parameters are $\epsilon_{\perp} = 45$ and $\epsilon_{\parallel} = 17.778$. As noted in the previous section, a slab with such dielectric parameters can be realized by alternating bars of commercially available barium titanate ($\epsilon \approx 80$) and alumina ($\epsilon \approx 10$) materials.
When the individual bar thicknesses are much smaller than wavelength, the stack exhibits the homogenized $\varepsilon_\perp$ and $\varepsilon_\parallel$ values. A thickness of 1mm for the anisotropic layers (having a misalignment of $\pi/4$) and 0.67mm for the free space layer results in a DBE at 10.6GHz.

To realize a highly directive antenna using small aperture size, DBE resonance and dielectric resonator antenna type lateral modes of the 3-D assembly must be utilized concurrently. Fig. 4.8(a) exemplifies such a design having lateral dimensions ($1.91 \times 1.91 \times 1.53$cm) optimized for directive radiation at 10.2GHz which is the closest Fabry–Perot transmission peak to the DBE. A small center-fed strip dipole (of size $0.2\lambda_0 \times 0.001\lambda_0$) is placed in the middle of the 3rd free space layer to examine directivity. Using the PMCHWT-SIE, we calculated that the single dipole achieves 8.16dB directivity as demonstrated in Fig. 4.8(b). We remark that the aperture is only $0.65\lambda_0 \times 0.65\lambda_0$ and thus the resulting aperture efficiency is 125.91%. This directivity increase is attributed to the DBE resonance. We also note that larger than 100% efficiency result is due considering only the top aperture’s physical size in the calculation. In reality, the side walls of the 3D structure also contribute to the directivity and must be taken into account.

Finally, to present the computational statistics associated with the numerical simulation of the DBE antenna, we consider the PMCHWT-SIE solution and FEM modeling of the same geometry in Ansoft’s HFSS. The corresponding directivity pattern obtained from FEM model is depicted in Fig. 4.8(c) and as seen agrees
well with the PMCHWT-SIE results. However, we note that the lateral dimension was slightly decreased from 1.91 cm to 1.88 cm in the FEM simulations, hence a slight frequency shift is observed between the PMCHWT-SIE and FEM results. To further decrease memory and CPU time requirement associated with the MoM system of the DBE assembly, we benefit from the crystal symmetries and employ FFT based matrix vector multiplications [84]. As seen in Table 4.1, this symmetric and repeating nature of the DBE assembly significantly decreases the storage requirements. More specifically, FEM required up to 3-4 GB of RAM to store the sparse matrix, while the PMCHWT-SIE formulation solved the same problem with a memory of less than 1 GB. However, there is certainly a tradeoff between the memory requirement and the computation time.

4.6 Remarks

The closed form dyadic Green’s function in the context of the PMCHWT-SIE provides a convenient and accurate analysis of high contrast uniaxial materials. Although
several formulations such as volume integral equation MoM and coupled dipole approximation method [70–72] are available for anisotropic media, the PMCHWT-SIE is much easier to implement in case of homogenous uniaxial materials. Moreover, the presented formulation can also be combined with the volume surface integral equation approach in [85] to provide a broader material modeling capabilities involving complex composites and metamaterials. Modeling of the uniaxial dielectrics with PMCHWT-SIE may be particularly useful in representing the uniaxial background permittivities of general inhomogeneous anisotropic materials.

The accuracy of PMCHWT-SIE was validated by comparing scattering patterns with the well known FE–BI method for various uniaxial media. In the radiation example, we found that slab resonances excited by the dipole are very sensitive to the orientation of the distinguished axis of the uniaxial slab. Our last example considered a small dipole radiator within a 3D DBE assembly. Specifically, we demonstrated that the 3D DBE crystal converts the small dipole into a directive radiator when its lateral dimensions are carefully designed.
CHAPTER 5

CHARACTERIZATION OF NATURAL AND ENGINEERED LOW LOSS UNIAXIAL DIELECTRIC MATERIALS AT MICROWAVE FREQUENCIES

Theoretical considerations of engineered metamaterials mostly assume lossless materials and use equivalent circuit models to demonstrate the validity of the underlying theory. However, when realistic material loss values are considered, especially for the resonant metamaterials (e.g. [10]) or slow modes of magnetic photonic (MPC) and degenerate band edge (DBE) crystals [31,60], the performance of the proposed antennas and devices may be severely affected. Therefore, realizing low loss metamaterials and being able to characterize their overall loss is critically important.

This chapter proposes a measurement methodology for characterizing low cost, engineered uniaxial crystals made from periodic dielectric assemblies to replace expensive natural crystals (such as sapphire, rutile or quartz). Although similar approaches for natural crystals have already been reported, this is the first time this method is adapted to engineered metamaterials. The measurement setup employs a highly resonant cavity providing high accuracy in characterizing low loss dielectrics at microwave frequencies [44–46]. To avoid depolarization, typically, a rod or disk shaped dielectric is placed in a cylindrical cavity to minimize the normal electric
field components at the dielectric–air interface. Although cylindrical geometries suppress depolarization, the more readily available rectangular samples are especially of interest for characterizing engineered uniaxial materials.

The proposed method [47] extends that in [45] to rectangular prism–shaped material samples. Further, it incorporates the finite element method (FEM) to determine resonant frequencies and mode field distributions within the sample loaded cavity. The sample permittivity is subsequently determined by matching the measured resonant frequencies to those computed via FEM. Loss tangent measurements with an accuracy of 4–5 significant digits can be achieved by this simple and effective measurement approach. Below, we first describe and validate the measurement approach for a natural uniaxial crystal, rutile (TiO$_2$). Afterwards, we characterize a layered barium titanate (BaTiO$_3$)–alumina (Al$_2$O$_3$) stack and show that low loss engineered crystals can indeed be achieved via a proper choice of a bonding agent. Finally, the use of these engineered uniaxial materials is demonstrated for a high gain 3D DBE antenna prototype [50].

5.1 Measurement Methodology and Rutile Characterization

Measurement methods for dielectric constant characterization can be classified in two groups: non-resonant and resonant methods. When using non-resonant methods, the permittivity of the dielectric sample is determined by solving an inverse problem to match the reflection and/or transmission coefficients to those measured [86–91]. Although, non-resonant techniques provide for wide band permittivity measurements, they are not sensitive enough for low loss characterization due to several reasons. For example, in waveguide–based setups, wall dissipation and machining requirements
to fit the sample into the waveguide cross section) reduce loss sensitivity [86]. Also, for the free space–based methods, the sample cross section should be large enough to minimize edge diffraction [86]. This requirement is likely too difficult to achieve, especially for engineered crystals and natural uniaxial crystals, due to manufacturing difficulties and cost. On the other hand, resonant methods utilize specific modes within the loaded cavities to estimate and distinguish dissipation on cavity walls from that within the sample. As a result, resonant methods provide more accurate dielectric constant and loss tangent characterization at the subject resonant frequencies. Among resonant methods, traditional perturbation techniques (see [92]) assume that the resonant mode distribution remains essentially the same after introduction of the dielectric sample within the cavity. Therefore, perturbation–based techniques are applicable for low to moderate permittivities (becoming less and less accurate as the permittivities of the samples increase to the point of modifying the mode distribution). Further, as noted, cavity wall dissipation decreases accuracy of the loss tangent.

Cavity perturbation techniques have indeed been extended to characterize uniaxial dielectrics [93] and combined with FEM to handle arbitrarily shaped and sized samples [94]. However, as mentioned above, this method still lacks the accuracy for measuring loss tangents. In contrast, cavities whose resonances allow for field concentration mostly within the sample under test are expected to be more accurate. An approach for designing such a measurement system was described in [45,95] for a cylindrical cavity geometry. This approach was also extended to characterize anisotropic samples in [46], again for cylindrical cavities and sample shapes. The same approach was further improved in [96] where FEM was also incorporated as
part of the measurement process. Recently, a highly resonant cylindrical cavity (with cylindrical shaped samples) was also employed to characterize anisotropic materials such as LCDs [97] and anisotropic dielectric sheets [98].

Nevertheless, cylindrical–shaped samples are often inconvenient, especially for engineered materials. In this paper, we extend the highly resonant cavity method to rectangular prism–shaped anisotropic material samples. As part of the permittivity tensor measurement process, we demonstrate an approach to select sample dimensions and their distinguished axis orientations. Further, use of FEM broadens the range of materials (e.g. textured dielectrics, artificial uniaxial dielectrics, etc.). For example, by using FEM, we are able to characterize (for the first time) an engineered uniaxial crystal assembly composed of layered dielectric materials. Below, we begin our discussion by presenting a step by step algorithm of the measurement procedure to first characterize a single crystal rutile ($\text{TiO}_2$) sample. This is followed by the characterization of a layered material sample with an equivalent uniaxial dielectric tensor.

5.1.1 Cavity Design and Measurement Procedure

1) Sample Choice: It is necessary to choose a cavity size much larger than the sample [45, 46] to excite the proper resonances (see Figs 5.1 and 5.2). Further, the sample permittivity needs to be roughly known. This information can be obtained from published data (e.g. [57] for rutile) or less accurate measurements (see [92]). As an example, for a uniaxial crystal, the relative dielectric tensor $\bar{\epsilon}$ has the form

$$\bar{\epsilon} = \epsilon_{\perp} \mathbf{I} + (\epsilon_{\parallel} - \epsilon_{\perp}) \hat{c} \hat{c}$$

(5.1)
Figure 5.1: The measurement setup and sample choices for rutile characterization

where $\epsilon_\parallel$ is the dielectric constant along the distinguished axis ($c$ or $\parallel$ axis) and $\epsilon_\perp$ is along the other directions ($a$ or $\perp$ axis) perpendicular to the $\parallel$ axis. For such uniaxial crystals, both $\epsilon_\parallel$ and $\epsilon_\perp$ must be known approximately before proceeding with the sample choice and cavity design. For rutile sample characterization, in our case, we used $\epsilon_\parallel = 165$ and $\epsilon_\perp = 85$ as previously reported in [57] for 5GHz. Characterization of uniaxial crystals requires the excitation of specific cavity modes such that $\epsilon_\perp$ and $\epsilon_\parallel$ are decoupled. To do so, one of the sample dimensions can be chosen smaller than the others, assuring that the lowest order resonant modes will only be polarized along the longer dimensions. Hence, when the $\parallel$ axis of the crystal is aligned with the shortest dimension, the lowest resonance characteristics (mode frequency, loss tangent, etc.) will only depend on $\epsilon_\perp$ and $\tan \delta_\perp$.

For the rutile sample of interest we chose the dimensions (see Fig. 5.1) $s_x = 3$, $s_y = 1.6$, and $s_z = 10$mm to be measured over 7–10GHz. Specifically, two rutile samples were obtained from MTI Corp (Richmond, CA). Of these, one had its $\parallel$ axis
Figure 5.2: Electric field components of the resonant mode concentrated within the sample having its $\parallel$ axis along $y$: (a) $|E_x|$, (b) $|E_y|$, (c) $|E_z|$. Note that $|E_y|$ ($\parallel$ axis direction) is essentially zero within the sample. Hence the resonance frequency mainly depends on $\epsilon_\perp$. As a result, the resonance frequency varied less than 1% for various $\epsilon_\parallel$ values.

along the $y$ axis and the second had the $\parallel$ axis along its $x$ axis. Fig. 5.1 depicts the overall measurement setup and the coordinate system used throughout the paper. To demonstrate the decoupling of $\epsilon_\perp$ from $\epsilon_\parallel$, we plot in Fig. 5.2 the field components for one of the lowest order resonant modes. In this case, the first sample was placed in a cavity whose dimensions were 8 times larger (initial guesses are used as permittivity values).

2) **Cavity Design:** The loss measurement accuracy is somewhat dependent on the cavity size. As can be understood, accuracy can be increased by reducing cavity wall losses. To observe this, we note that in absence of radiation, the cavity $Q$ has the form \[99\]

$$Q_u^{-1} = p_{es}^\perp \tan \delta_\perp + p_{es}^\parallel \tan \delta_\parallel + R_s/G,$$

(5.2)

where $Q_u$ is the ratio of average stored energy to that dissipated within a cycle (when the sample loaded cavity is not connected to any other external circuitry); $p_{es}^\perp/p_{es}^\parallel$ are the electric field filling ratios along the $\perp/\parallel$ axes, and $\tan \delta_\perp/\tan \delta_\parallel$ are the loss tangents associated with $\epsilon_\perp/\epsilon_\parallel$. Further, $R_s$ is the cavity wall surface resistivity, and
$G$ is a geometry factor [45] given by

$$G = \frac{\omega \int \int \int_V \mu_0 \mathbf{H} \cdot \mathbf{H}^* dv}{\int \int_S \mathbf{H}_{\text{tan}} \cdot \mathbf{H}_{\text{tan}}^* ds}.$$  \hspace{1cm} (5.3)

As usual, $\mathbf{H}$ refers to the magnetic field (with $\mathbf{H}_{\text{tan}}$ being the tangential magnetic field on the walls), and $\mu_0$ is the free space permeability. Thus, $G$ represents the ratio of the stored energy within the loaded cavity to that dissipated on the cavity walls.

In practice, at least 10% of the overall energy loss/dissipation should be due to the sample under test for accurate loss measurements. Hence, from (5.2), a cavity having large $G$ (e.g. $\sim 10,000$) is preferred since it minimizes the last term as compared to the others.

To evaluate $G$ and the resonant mode $\mathbf{H}$ fields, the eigenvalues (resonant frequencies) and eigenvectors of the sample loaded cavity system

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r}) - k_0^2 \mathbf{\varepsilon}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r}) = 0, \quad \mathbf{\varepsilon}(\mathbf{r}) = \begin{cases} \mathbf{\varepsilon}_r, & \mathbf{r} \in \text{uniaxial sample} \\ \mathbf{I}, & \mathbf{r} \in \text{cavity} \end{cases}$$  \hspace{1cm} (5.4)

must be determined. Here, $\mathbf{E}$ denotes the electric field inside the cavity, $k_0$ is the free space wavenumber, $\mathbf{\varepsilon}_r$ is the relative dielectric constant tensor of the uniaxial sample, and $\mathbf{I}$ is the identity dyad. An FEM solution of (5.4) provides the resonant frequencies and the discrete form of (5.3) to numerically evaluate $G$ (Appendix C).

Using FEM analysis and initial permittivity guesses (Step 1), dependence of $G$ on cavity size for the two lowest order resonances is depicted in Fig. 5.3(a). The plot shows the value of $G$ as the cavity size is increased from 3 to 14 times of the sample size. It is seen that $G$ reaches a maximum when the cavity size is about 10 times the sample, implying that the resonant fields are almost totally concentrated inside the sample for this setup. Nevertheless, for practical purposes, a copper cavity 8 times the size of the sample, providing a $G \sim 10,000$ for the lowest order resonance (higher
order resonances tend to have even larger $G$ factors), was manufactured. That is, the cavity used in the setup (shown in Fig. 5.3(b)) had dimensions $c_x = 24$, $c_y = 12.8$ and $c_z = 80$mm. Noting that the resistivity of copper is $R_s \approx 30$ mΩ at 10GHz, and losses within the dielectric sample should be at least 1/10th of the overall loss, we conclude from (5.2) that the loss tangent sensitivity ($\Delta \tan \delta$) of the setup will be $\Delta \tan \delta = 3 \times 10^{-7}$ for $G = 10,000$. On the other hand, if the cavity wall losses are not accounted for, the sensitivity would have been $\Delta \tan \delta = 3 \times 10^{-5}$.

3) Cavity Excitation/Feed: To excite the resonant modes, a short coaxial probe can be used as the feed. For the best excitation, the probe location must be chosen to match the highest normal electric field value at the cavity wall parallel to the $xy$ plane (see Fig. 5.1). For rutile characterization we specifically designed two cavity covers with different probe locations to excite different modes. Depending on the mode field distributions at the cavity walls, two different probe locations helped us distinguish and classify higher order resonances. One of the probe location was selected at the middle of the $xy$ wall, and the other was 12mm away from the first probe location along the $z$ axis as shown in Fig. 5.1.
Steps 1–3 finalize the cavity choice, sample size and excitation. These choices are critical to achieving high loss sensitivity. Sample size and orientation permit decoupling \( \parallel \) and \( \perp \) axis properties.

4) **Extraction of** \( \epsilon_{\perp} \) **and** \( \epsilon_{\parallel} \): To determine \( \epsilon_{\perp} \), the first sample is placed in the cavity so that its \( \parallel \) axis coincides with the \( \hat{y} \) direction. The cavity was then excited by connecting the probe feed (see Fig. 5.1) to the network analyzer (Agilent E8362B, 10MHz–20GHz PNA Series Network Analyzer). The resonant frequencies were then determined by finding the null locations in the measured reflection coefficient curve (\( |S_{11}| \)). These frequencies are given in Table 5.1 (first column) for the rutile sample.

Having obtained the measured resonant frequencies, a one dimensional search for \( \epsilon_{\perp} \) was then performed using the FEM analysis till the measured frequencies matched the calculated ones. We note here that the FEM readily facilitates several runs for \( \epsilon_{\perp} \) very efficiently. For the specific rutile sample, the calculated frequencies for different \( \epsilon_{\perp} \) values are given in Table 5.1. From these we deduce that \( \epsilon_{\perp} = 89 \) predicted the resonances within 1.6\% of the experimental values. Thus, it was chosen as the best approximation for \( \epsilon_{\perp} \) of the rutile sample.

We could next excite higher order modes polarized along the \( \parallel \) and \( \perp \) axes to determine \( \epsilon_{\parallel} \). However, this may lead to longer CPU times in FEM simulations due to increased resonant frequencies requiring finer FEM discretization. Therefore, to determine \( \epsilon_{\parallel} \), a second sample having the \( \parallel \) axis along the \( \hat{x} \) direction was placed inside the cavity. The cavity resonances were again determined from the \( |S_{11}| \) measurement.

Similar to the procedure for \( \epsilon_{\perp} \), a simple numerical search to match the resonant frequencies with the measured ones lead to \( \epsilon_{\parallel} = 168 \) (see Table 5.2, column 3). More
specifically, for the case of $\epsilon_\parallel = 168$, all simulated resonant frequencies are within 1.0% of the experimental. We remark that the small errors (1.6% for $\epsilon_\perp$ and 1.0% for $\epsilon_\parallel$) (in Table 5.1 and 5.2) between measured and simulated resonance frequencies can be improved by increasing FEM mesh density at the expense of CPU time. However, depending on the uncertainty of the sample location, a finer FEM mesh may not appreciably increase accuracy for $\epsilon_\perp$ and $\epsilon_\parallel$. On the other hand, the relative accuracy of the loss tangent measurement primarily depends on the $G$ factor magnitude.

5) **Loss tangent extraction**: The last step in the characterization of rutile is the determination of the loss tangents $\tan \delta_\perp$ and $\tan \delta_\parallel$ along the $\perp$ and $\parallel$ axes, respectively. To do so, $Q_u$, $p^\perp_{es}$, $p^\parallel_{es}$, and $G$ in (5.2) must be calculated. As such, one final FEM simulation must be performed using the permittivity values obtained from Step 4 (i.e. $\epsilon_\perp = 89$ and $\epsilon_\parallel = 168$) to determine $p^\perp_{es}$ and $G$.

The electric filling ratios are found by integrating the field solutions using

$$p^\perp_{es} = \frac{W^\perp_{es}}{W_{et}} = \frac{\int \int \int_V \epsilon_\perp \parallel E^\perp \parallel \cdot \mathbf{E}^* \parallel dV}{\int \int \int_V \mathbf{E} \cdot \mathbf{\tau}(r) \cdot \mathbf{E}^* dV}. \quad (5.5)$$
As usual, $W_{\perp es}$ and $W_{\parallel es}$ are the stored electric energies along the $\perp$ and $\parallel$ axes, respectively, and $W_{et}$ is the total stored electric energy inside the sample loaded cavity. For calculating $G$, we refer to (5.3).

If the calculated $G$ factors are smaller than the required (to satisfy the desired measurement sensitivity, (see Step 3), the overall cavity design process must be repeated with the measured permittivity values to increase the $G$ factor (Steps 1–3).

For our case, the loss tangents can be determined accurately at the well matched ($|S_{11}| < -20$ dB) resonant frequency of 9.78GHz. Having calculated the quality factors ($Q_u$) from the $S_{11}$ measurements (as in [99]), electric field filling ratios and $G$ factor from FEM simulation (as in Step 3 and 4), (5.2) can be subsequently used to determine $\tan \delta_{\perp,\parallel}$. From (5.2), we calculated $\tan \delta_{\perp} = 1.39 \times 10^{-4}$ and $\tan \delta_{\parallel} = 8.6 \times 10^{-5}$ at 9.78GHz. Although these values are higher than those reported in [57] (around 5GHz), we should note that our frequency is also higher. Also, in contrast to [57], the loss tangent along the $\perp$ axis was found to be larger than that along the $\parallel$ axis. We attribute this to different impurities within the samples.

As can be understood, the above steps can be readily adapted to characterizations of other materials, including engineered periodic assemblies. In summary, Steps 1–3 determine the cavity and sample sizes, whereas Step 4 is used to extract the permittivities associated with the $\perp$ and $\parallel$ axes. Finally, Step 5 uses these permittivity values in a final FEM simulation to determine the associated loss tangents.
5.2 Loss Characterization of Periodic Assemblies

An increasingly important application of the proposed measurement setup is the characterization of engineered materials, such as periodic structures. For low frequencies (where the periodicity is much smaller than the wavelength within the material), the structures can be characterized by an effective permittivity tensor. In this case, the above procedure can be used. However, for layered and periodic media, the FEM analysis plays an important part of the characterization process. As an example, let us consider a stack of layered alumina–barium titanate (Al$_2$O$_3$–BaTiO$_3$) platelets. This assembly (see Fig. 5.4) emulates an artificial uniaxial dielectric medium [49]. An important aspect of this sample is the necessity to model the assembly using the individual layer dielectric constants to verify the validity of the equivalent model at the frequency range of interest. Using manufacturer supplied values of $\epsilon = 82$ for titanate and $\epsilon = 9.6$ for alumina, we obtain the equivalent permittivity tensor values of $\epsilon_{\perp} = 45.8$ and $\epsilon_{\parallel} = 17.2$ (see [49]). We actually found rather good agreement between the resonant mode distributions and frequencies (within 2-3%) when the FEM analysis was carried out using the individual layer dielectric constants and their equivalent tensor. This was somewhat expected since the individual layer thicknesses (e.g. 0.508mm = $\lambda_g$/8.6 at the resonant frequency of 7.54GHz with the first sample (Fig. 5.4(a)) in the cavity) are much smaller than the wavelength within the high contrast titanate layers ($\lambda_g$). Fig. 5.5 presents a comparison of the resonant field distribution within the equivalent media and the actual layered medium shown in Fig. 5.4(a). Thus, we can use the equivalent homogenized tensor (instead of the individual layers) to extract the permittivities and loss tangents with greater ease (i.e. 77
less CPU time by using the homogenized model). Taking the above dielectric constants as initial values for the permittivity tensor, two samples of size \( s_x = 7.6 \), \( s_y = 4 \) and \( s_z = 25 \)mm were prepared for the cavity shown in Fig 5.3(b) giving a \( G \) factor of \( \sim 1,000 \). The samples consisting of 15 and 8 layers (with alternating alumina and barium titanate layers each of 0.508mm thickness) are shown in Fig. 5.4(a) and (b) respectively. The 15-layered stack has its \( \parallel \) axis along \( x \) direction, and the 8 layered stack has \( \parallel \) axis along the \( y \) direction. As already discussed, the value of \( G \sim 1,000 \) implies an accuracy of \( \Delta \tan \delta = 1 \times 10^{-4} \). To extract the equivalent tensors of these stacks, the employed FEM analysis used 22400 hexahedra with 3000 in the stack.
This tessellation was more than sufficient since a coarser mesh (7440 hexahedra with 800 in material sample) predicted a resonance that was within 1%.

Inevitably, such ceramic structures are prepared using a suitable low loss glue to assemble the individual layers together. However, the bonding agent between the layers can significantly increase the overall loss of the engineered material. Using the measurement procedure detailed in section 5.1, we extracted the loss tangents of the layered structure assembled with different low loss glues. We found that the measured resonances were within 1% of those numerically computed with equivalent permittivities $\epsilon_\perp = 45.8$ and $\epsilon_\parallel = 17.2$. Thus, step 4 was skipped. Following the procedure in step 5, we used the second sample shown in Fig. 5.4(b) to extract the loss tangent of $\tan\delta_\perp = 8 \times 10^{-4}$ at 7.57GHz for the unglued layers (held together by a tiny plastic strap). Using the sample in Fig. 5.4(a), we also identified a loss tangent of $\tan\delta_\parallel = 6 \times 10^{-4}$ at 6.95GHz. However, when the M Bond 610 (a two component phenol-epoxy) glue is used to hold the assembly together, $\tan\delta_\parallel$ increased significantly up to $\tan\delta_\parallel = 3.8 \times 10^{-3}$, whereas $\tan\delta_\perp = 1 \times 10^{-3}$ was very close to the value without the glue. This increase in loss tangent is certainly a major concern. However, even if it is an order of magnitude larger than that of rutile, the loss is still acceptable for thin slow wave resonant structures such as DBE crystals.

Fig. 5.6 shows a DBE assembly of 3 unit cells, each consisting of two 1mm thick misaligned engineered uniaxial layers (prepared with M Bond 610) with 0.25mm air gap. The optimum rotation angle was determined to be 55°, estimated from FEM simulated gain patterns, and confirmed by experimental patterns. Fig. 5.7 shows simulated and measured far field gain patterns of the antenna assembly when fed by a slot coupled microstrip line (not shown). These constitute the first measurements...
Figure 5.6: DBE assembly composed of three repeating units: (left) top view and (right) side view. The transparent spacers are 1.25mm high quartz tubes.

Figure 5.7: Measured and simulated gain (dB) patterns of the DBE prototype: (left) $xz$ (right) $yz$ plane

of a prototype antenna, based on the extraordinary propagation modes supported by DBE crystals. Disagreement between the measurements and FEM simulations can be attributed to dielectric losses in the feed structure and the uniaxial layers, as well as the external foam used to hold the sample in place during anechoic chamber measurements. Details on the fabrication of the engineered uniaxial layers and their loss dependence to various glue types can be found in [50]. On the other hand, [100] presents the overall design of the DBE antenna prototype in Fig. 5.6 in terms of layer thicknesses, misalignment angle, aperture size, and feeding mechanism.
5.3 Remarks

This chapter presented a simple methodology to characterize low loss engineered materials and natural uniaxial dielectrics. The measurement setup was a rectangular prism–shaped cavity coupled with the FEM to accurately determine the permittivity tensor and loss tangent of uniaxial materials. We demonstrated an FEM–based search as an integral part of the characterization process. Also, effective parameters for engineered layered metamaterials were characterized for the first time. For rutile (at room temperature), we found that the crystal exhibits relative dielectric constants of $\epsilon_\perp = 89$ and $\epsilon_\parallel = 168$ with loss tangents $\tan \delta_\perp = 1.39 \times 10^{-4}$ and $\tan \delta_\parallel = 1.09 \times 10^{-4}$ at 9.78GHz. We fabricated and measured an engineered uniaxial medium built by combining barium titanate and alumina layers. For this sample, we measured a low loss tangent of $\tan \delta_\perp = 8 \times 10^{-4}$ and $\tan \delta_\parallel = 6 \times 10^{-4}$ at about 7GHz when no glue is used to hold the stack together. In addition, a 3D DBE antenna prototype with high radiation efficiency was realized using these engineered uniaxial layers. Although we concentrated on rutile and layered stacks, the same measurement procedure can be applied to characterize low loss samples for any class of anisotropic bulk or periodic materials (for frequencies where homogenization applies) where high accuracy maybe required.
CHAPTER 6

MINIATURE PRINTED ANTENNA DESIGN VIA PARTIALLY COUPLED TRANSMISSION LINES EMULATING DEGENERATE BAND EDGE CRYSTALS

In the previous chapters, we demonstrated how higher order dispersion (K-ω) diagrams and the associated propagation modes of magnetic photonic (MPC) and degenerate band edge (DBE) crystals can be harnessed to realize directive volumetric antennas. As Chapter 5 mentioned, the first DBE crystal prototype was a volumetric structure composed of engineered anisotropic dielectric layers. However, it is also possible to employ coupled pairs of printed transmission lines to emulate anisotropy needed to realize DBE properties [51] on standard microwave substrates. In turn, such printed circuit emulation of the DBE modes provide for a straightforward approach to realize small antennas harnessing this mode.

This chapter proposes a miniature printed antenna design approach [53] that exploits higher-order dispersion behavior in DBE crystals. First, the benefit of DBE mode for printed antenna miniaturization is explained through the dispersion diagram point of view. Following this, a printed DBE microstrip (MS) line layout and a method to design and tune this circuit to realize a DBE resonant structure is presented. Subsequently, a resonant antenna is formed by cascading two such MS-DBE
unit cells in a circularly periodic fashion. The first part of the chapter is devoted to the antenna/array design and characterization using a low contrast dielectric substrate (Rogers Duroid, $\epsilon_r = 2.2$). Next, fabrication and design verification using a high dielectric constant substrate (alumina: $\text{Al}_2\text{O}_3$, $\epsilon_r = 9.6$) is given. The manufactured MS-DBE antenna was measured to have a broadside gain of 4.5dB at 1.48GHz with 3.0% bandwidth. For this gain and bandwidth, we demonstrate that the proposed antenna has a $\lambda_0/9 \times \lambda_0/9$ footprint and is among the smallest in the literature.

6.1 Printed Antenna Miniaturization Using DBE Dispersion

To reduce antenna size, a goal is to shift the resonance to lower frequencies. For miniaturization, it is therefore essential to have an understanding of the resonance conditions. As an example, consider the printed loop structure shown in Fig. 6.1(a) and its dispersion diagram in Fig. 6.1(b). This diagram (which implies an inherent periodicity [13]) was generated by using half of the loop as the unit cell to build the periodic structure. As shown, the loop is formed by a circularly periodic structure involving two unit cells (top and bottom half loops). The corresponding band diagram in Fig. 6.1(b) indicates two supported propagating waves (two traces) corresponding to $K < 0$ and $K > 0$. Resonance occurs at $K = \pm \pi$ and $K = \pi$, since these points are associated with matching phases of the two propagating waves on the loop. Of particular interest is the resonance at $K = \pi$, and to reduce its frequency of occurrence, it is necessary to lower the $K-\omega$ curve as shown in Fig. 6.1(c). This can be done by inserting reactive elements within the transmission lines. In general, such a periodic structure leads to a 2nd order $K-\omega$ regular band edge (RBE) curve at about $K = \pi$ frequencies. Although the frequency at $K = \pi$ can be further pushed
Figure 6.1: (a) Simple printed loop antenna. (b) Dispersion diagram of a unit cell forming the rectangular printed loop antenna; resonances of this circularly periodic structure (2 unit cells) are marked by dots. (c) Bending K-ω diagram to shift resonance to lower frequencies. (d) Magnified view of the dispersion diagram around the band edge

down by increasing loading to lower the frequency, the parabolic relation of the K-ω curve still remains and limits the amount of curvature at band edges. However, DBE crystals exhibit 4th order K-ω curves at band edges. Having such a relation near the band edge implies maximally flat K-ω curves at K = π. As seen in Fig. 6.1(d), a maximally flat K-ω curve pushes the K = π points further down in frequency, thus improving miniaturization. Also, the stronger resonance at K = π can lead to better impedance matching for an antenna resonating at the DBE mode.

6.2 DBE Unit Cell Using Printed Coupled Lines

The simple volumetric DBE unit cell is composed of two anisotropic homogenous layers having different permittivity tensors (see Fig. 6.2(a)). A careful choice of layer
thickeneses and misalignment angle between successive anisotropic layers results in a periodic assembly that exhibits considerably flatter band edge [29]. This K-ω behavior can be realized using a pair of transmission lines on a uniform substrate as depicted in Fig. 6.2(b). Each transmission line is thought as carrying one polarization component of the electric field propagating within the DBE crystal. By using different line lengths a phase delay is introduced between the two polarizations to emulate diagonal anisotropy. In a similar fashion, even–odd mode impedances and propagation constants on the coupled lines can be used to emulate a general anisotropic medium (i.e. non-diagonal anisotropy tensor). In any case, by cascading uncoupled and coupled transmission line sections as in Fig. 6.2(b), an equivalent printed circuit is realized that emulates the volumetric DBE crystal. The circuit is then tuned to achieve the DBE mode by appropriately selecting the line thicknesses and coupled line separations.

To harness the maximally flat DBE band diagram for reduced size antennas, we propose the modified unit cell layout in Fig. 6.3. The unit cell realizes circular periodicity by having all ports on its same side. As seen, the transmission line sections consist of longer uncoupled lines (see Fig. 6.2) bent towards the center of the structure.
Figure 6.3: DBE unit cell on a 100mil thick Duroid substrate and its corresponding band diagram
to keep the footprint as small as possible. The dispersion diagram of this structure is computed using the transfer matrix approach [31] once the ABCD matrix [92] of the unit cell has been extracted. Since many parameters play an important role in realizing the DBE mode (line thicknesses and separations, uncoupled and coupled line lengths, etc.), a two step algorithm was used to speed up the design process. At first, we extracted the ABCD matrices from a transmission line analysis. For that, the fast microstrip line simulator module in the ADS (Advanced Design Systems) software was used. The DBE cell parameters were then designed via trial-error and fine tuned via a full wave electromagnetic solver. In this paper, we used HFSS v10 from ANSOFT Corp. as our full wave solver. Fig. 6.3 gives the dimensions and dispersion diagram of a DBE unit cell designed on a 100mil thick Rogers Duroid substrate having $\epsilon_r = 2.2$ with a low loss tangent of $\tan \delta = 9 \times 10^{-4}$. Although the band diagram is not very flat at $K = \pi$, all four evanescent branches in the bandgap meet at the band edge, implying a DBE mode at 2.6GHz. However, the band diagram can be made flatter at $K = \pi$ by fine tuning the circuit geometry (e. g. by slightly decreasing the uncoupled line thickness). Nevertheless, we avoided such circuit tuning since practical limits set by the printed circuit manufacturers do not allow us to realize this tuning. In the
following, we present an analysis of the MS-DBE antenna built from the unit cell shown in Fig. 6.3.

6.3 MS-DBE Antenna Performance

The geometrical layout and performance of the MS-DBE antenna composed of two circularly cascaded unit cells is depicted in Fig. 6.4(a). As designed, the antenna has a strong resonance at 2.59GHz (i.e. very close to the DBE frequency of 2.6GHz). For impedance matching the MS-DBE antenna to 50Ω, a capacitively coupled coaxial cable was used as depicted in Fig. 6.4(a). This feeding mechanism provided an excellent match at the DBE frequency with $|S_{11}| \sim -20$dB. From Fig. 6.4(b), we observe that the antenna radiation mechanism is very similar to a patch except for the high field concentration at the antenna center. The latter may be exploited to form a tightly packed antenna array, since inter-element field couplings will be minimal. As depicted in Fig. 6.4(c) and (d), the antenna has a rather large 6.9dB broadside gain at 2.59GHz with 0.8% bandwidth (defined by $|S_{11}| < -10$dB), and a physical footprint of $\lambda_0/4.3 \times \lambda_0/5$. The radiated field is linearly polarized and the efficiency is greater than 95%. This high level of efficiency is attributed to the very low loss Duroid substrate.

Although miniaturization can be achieved by moving the $K = \pi$ resonance to lower frequencies, a maximally flat $K-\omega$ curve implies an extremely dispersive structure. Thus, the MS-DBE antenna suffers from a small bandwidth. Nevertheless, once the footprint is designed, the substrate thickness can be increased to attain higher bandwidth. The higher bandwidth with thicker substrates can be easily observed from the band diagrams depicted in Fig. 6.5. As seen, increasing substrate thickness from
Figure 6.4: (a) MS-DBE layout on a $2'' \times 2''$, 100mil thick Duroid substrate having $\epsilon_r = 2.2$, $\tan \delta = 9 \times 10^{-4}$ (b) Tangential electric field amplitude on the top surface of $2'' \times 2''$ substrate (c) $|S_{11}| < -10$dB bandwidth of the MS-DBE antenna (d) Radiation pattern of the MS-DBE antenna at 2.59GHz.

100mil to 250mil decreases the band gap width. Thus, the upper branch of the $K\omega$ curve approaches the lower branch causing mode coalescence. Also, since a higher sloped $K\omega$ curve implies smaller dispersion, the antenna bandwidth is increased to 3.3% (from a mere 0.8%). When a 500mil thick substrate is used (instead of 250mil), we observe an even larger bandwidth of 8.7%. However for this case, the resonance frequency is lowered to 2.3GHz (as compared to 2.59GHz for the 100mil thick substrate). The gain at this lower frequency is nearly 6.2dB being only 0.7dB lower than that of the 100mil thick antenna. Therefore, once the MS-DBE antenna footprint is designed, the substrate thickness may be used to adjust the bandwidth without compromising other antenna parameters such as electrical size and gain. Increasing the substrate thickness beyond a certain level will eventually change the
Figure 6.5: Substrate thickness effects on the MS-DBE antenna performance. Thicker substrates enhance the bandwidth by narrowing the band gap at the K-\(\omega\) diagram.

As mentioned above, the high-field concentration at the center of the antenna make the MS-DBE suited for tightly packed antenna arrays. To demonstrate this, we consider the array configuration depicted in Fig. 6.6. In this, the antenna elements are separated 300mil from each other, equivalent to an inter-element spacing of \(\lambda_0/15\) at the center frequency of 2.55GHz. The radiation efficiency can be further increased by rotating the adjacent elements 90° with respect to each other as shown in Fig. 6.6. Each element in the 4 \(\times\) 4 array is fed using a capacitively coupled 50Ω coaxial cable, and by adjusting the phase shifts between adjacent elements, circular polarization can be readily achieved. The lateral size of the array is 5.5′′ \(\times\) 5.5′′, equivalent to an
Figure 6.6: MS-DBE antenna array using the element in Fig. 6.4. The array size is 5.5'' × 5.5'' and printed on a 250mil thick Duroid substrate. Element spacing is 300mil and 90° rotation of adjacent elements achieves dual polarization. The array has an electrical size of 1.2λ₀ × 1.2λ₀ × λ₀/18 at 2.55GHz with 2% bandwidth and 11.7dB (realized) gain.

aperture of 1.2λ₀ × 1.2λ₀ at 2.55GHz. Specifically, the array has 13dB directivity and 2% bandwidth on a 250mil thick Rogers Duroid substrate. The main beam can be scanned down to 80° in the y–z plane and down to 70° in the x–z plane. Considering the top surface (ignoring sidewalls), the MS-DBE array has 111% aperture efficiency as compared to the optimal 12.6dB gain for the same size uniform aperture. When the 75% radiation efficiency is included (25% is lost to mismatches and inter-element coupling), the realized gain of the array drops to 11.7dB corresponding to 80% realized aperture efficiency. It should be noted that the presented MS-DBE array is not an optimum design, and is merely presented to demonstrate the advantages of the DBE element for conformal applications.

From the above, we summarize that the two unit cell printed antenna (harnessing the DBE resonance) can be designed from its dispersion diagram. Moreover, it can
be realized on commercially available substrates using standard printed circuit technology. In the following, we focus on further miniaturization of the MS-DBE antenna by increasing the substrate dielectric constant.

### 6.4 Miniature MS-DBE Antenna on Low-Loss Alumina Substrate

To further reduce the size of the MS-DBE antenna we proceeded to employ a high contrast substrate. In general, substrate losses (as well as capacitor and inductor losses) become critical to the performance of metamaterial-based antennas. For example, losses in miniature $\lambda_0/10 \times \lambda_0/10$ resonant antennas built from left handed metamaterials \cite{25, 26} exhibited efficiencies well below 60\% due to capacitor and inductor loadings. Therefore, miniaturization of an MS-DBE antenna without decrease in radiation efficiency requires careful substrate choices to reduce losses. Among readily available high contrast dielectric materials, we chose $\text{Al}_2\text{O}_3$ having $\epsilon_r = 9.6$, $\tan \delta = 3 \times 10^{-4}$. To design the MS-DBE antenna, we followed the method described in Section II using an initial 100mil thick alumina substrate. The final MS-DBE unit cell had the same line thicknesses, lengths, and coupled line separation. Only the thickness of the outmost transmission line (originally 125mil thick) was replaced with a 110 thick line. We should note that the resulting band diagram is very similar to the low contrast substrate $\omega$ curves shown in Fig. 6.5. However, the DBE occurs at a lower frequency (1.4GHz) and the band gap is smaller due to the higher dielectric constant.

Having completed the unit cell design, we proceeded to build a two unit cell MS-DBE antenna. As expected, due to the higher dielectric constant substrate, the MS-DBE antenna had a very small bandwidth (much less than 1\%). Thus, we increased
the substrate thickness to 500mil without changing the antenna layout. Since the band diagram changes in much the same way as that of the low contrast substrate (see Fig. 6.5), the resulting antenna bandwidth increases up to 2%. However, we further observed that decreasing the outer line thickness resulted in wider bandwidths. This benefit, though, comes at the expense of a larger footprint size due to the increase of the resonant frequency. Since we desired to utilize the layout area better with a square shaped footprint, we decided to modify the MS-DBE antenna by decreasing the outer line thickness to 25mils. Although we still observed an increase in the electrical size, this final design had 3.5% bandwidth.

Fig. 6.7(a) depicts the layout of the final MS-DBE antenna design. The outer lines are considerably thinner than the others. The antenna is again fed through a capacitively coupled 50Ω coaxial cable realizing a probe positioned 130mil away from the edge of the inner vertical line. This design occupies a 0.88″ × 0.85″ footprint on a 2″ × 2″, and 500mil thick alumina substrate. As usual, the back surface of the substrate is metal coated to form the ground plane. Although the physical size of both antennas is about the same on high and low contrast substrates, the MS-DBE element on alumina substrate (\(\epsilon_r = 9.6\)) resonates at 1.445GHz as opposed to 2.59GHz for the Duroid substrate (\(\epsilon_r = 2.2\)). Therefore, the electrical footprint is as small as \(\lambda_0/9.6 \times \lambda_0/9.3\) on the alumina substrate. The mode structure is concentrated around the footprint as shown in Fig. 6.7(b). As depicted in Fig. 6.7(c) and (d), this element has a gain of 4.25dB at broadside with 3.5% bandwidth. Moreover, the low loss alumina substrate implies a radiation efficiency greater than 95%.

Clearly, the designed MS-DBE antenna is very attractive due to its small print size, high radiation efficiency, reasonable bandwidth and broadside gain. To verify
Figure 6.7: (a) MS-DBE layout on a 2'' × 2'', 500mil thick substrate having ε_r = 9.6, tan δ = 3 × 10^{-4}; the footprint size is λ_0/9.6 × λ_0/9.3 × λ_0/16 at resonant frequency of 1.445GHz (b) Tangential electric field magnitude on the top surface of 2'' × 2'' substrate (c) |S_{11}| < −10dB bandwidth of the MS-DBE antenna (d) Radiation pattern of MS-DBE antenna at 1.445GHz. Radiation efficiency is > 95%

the design, we proceeded to manufacture it using 2'' × 2'' AD–995 substrate platelets (from CoorsTek Inc). The AD–995 substrate is 99.5% pure alumina with electrical properties ε_r = 9.7 and tan δ = 1 × 10^{-4}. Since the thickest readily available AD–995 platelet is only 50mil thick, we ordered 10 such laminates and stacked them up to realize the 500mil thick substrate. Each laminate had an identical 50mil diameter hole to support the coaxial probe and only the top and bottom laminates contained the metallizations. A silver alloy was used for the conducting material with a 10mil pull–back applied to the edges of the ground plane due to manufacturing limitations.
To hold the individual alumina platelets together, two plastic straps were used instead of glue to minimize losses. We tightened these straps as much as possible to avoid possible air gaps between each alumina laminate. The bandwidth and gain of this prototype was measured in the Ohio State University, ElectroScience Laboratory compact range. The measured gain and bandwidth plots, given in Fig. 6.9, show a good agreement with the calculated curves. Specifically, we measured a 4.5 dB broadside gain with 3.0% bandwidth. The main lobe of the radiation pattern also agrees very well with the computations. Only the resonance frequency slightly shifted from 1.445 GHz to 1.48 GHz. There is also a small change in the bandwidth from 3.5% to 3.0%. These small variations may very well be related to manufacturing
uncertainties and the feed location. The antenna footprint is $\lambda_0/9 \times \lambda_0/9 \times \lambda_0/16$ at 1.48GHz. In any case, the measured 4.5dB gain implies a radiation efficiency as high as 95%.

The small electrical size and large radiation efficiency place this MS-DBE antenna among the smallest in the recently reported literature. Although other metamaterial antennas [25, 26] achieved a small electrical size (on the order of $\lambda_0/10 \times \lambda_0/10$), the included capacitive and inductive loadings resulted in radiation efficiencies of less than 60%. Specifically, the 0th order planar patch in [25] has a very small bandwidth and the miniature antenna in [26] suffers from small bandwidth and a low 44% radiation efficiency. On the other hand, although a radiation efficiency of 97% was reported for the metamaterial-inspired small antennas in [24], these designs had very small bandwidth. To better evaluate this MS-DBE antenna below we consider comparisons of gain × bandwidth products.

To make a more fair comparison, we consider the footprint size of our antenna and the one in [26]. This choice is well justified when the field distributions around the conductors (as shown in Fig. 6.7(b)) is considered. Also, since the antennas in [24] were built on large ground planes, we will ignore the ground plane size. To carry out the comparison, the $|S_{11}| < -10$dB fractional bandwidth (FBW) was used for the MS-DBE element, spherical folded helix in [101], and metamaterial based antenna in [26]; however, for the metamaterial inspired antennas in [24] $|S_{11}| < -3$dB FBW was used. These values were then employed to compute the antenna quality factor $Q$ from $\text{FBW} = (\text{VSWR} - 1)/(Q \sqrt{\text{VSWR}})$ [2], where VSWR is the voltage standing wave ratio corresponding to the particular $|S_{11}|$ value defining the FBW. As the $Q$ value is independent of bandwidth, we proceeded to plot the $G/Q$ ratio vs. $ka$ (instead
Figure 6.10: Comparison of gain × bandwidth products of the MS-DBE antenna in Fig. 6.8 of gain ($G \times$ BW vs. $ka$) for a more fair comparison of the various antennas. Fig. 6.10 depicts the optimal $G/Q$ calculated in accordance to [1,102]. However, since small antennas can exceed the gain limit defined in [102], another curve with a fixed gain of 3 is also provided. As seen, the spherical folded helix [101] performs best in comparison to other planar antennas by utilizing the spherical volume. The MS-DBE antenna provides the largest $G/Q$ ratio among all other metamaterial based antennas [24,26]. Specifically, the footprint size of the MS-DBE antenna is almost on the $G/Q$ limit being only 1.06 times larger than that required to achieve the optimal.

6.5 Remarks

This chapter harnessed maximally flat DBE dispersion diagrams for the first time to design and realize miniature printed antennas. By using a two-step design process (consisting of a fast transmission line simulator and a full wave electromagnetic solver), we combined coupled and uncoupled lines to realize a DBE unit cell on a
uniform substrate. A resonant MS-DBE antenna was formed by cascading two such unit cells in a circularly periodic fashion. The MS-DBE bandwidth was then tuned to improve bandwidth. The manufactured MS-DBE antenna on alumina substrate had a gain of 4.5dB at 1.48GHz with 3.0% bandwidth. It was also shown that this MS-DBE antenna was among the smallest metamaterial antennas having a $\lambda_0/9 \times \lambda_0/9$ footprint and a gain×bandwidth product that exceeded recently presented metamaterial antennas. A high gain, dual-polarized array configuration on low contrast substrate was also proposed. This $4 \times 4$ MS-DBE array had a $1.2\lambda_0 \times 1.2\lambda_0$ aperture and achieved 111% aperture and 75% radiation efficiency.
CHAPTER 7

LUMPED CIRCUIT MODELS FOR DEGENERATE BAND EDGE AND MAGNETIC PHOTONIC CRYSTAL DISPERSION DIAGRAMS

The behavior of a $K-\omega$ curve can conveniently be approximated with a polynomial relation around slow group velocity frequencies (i.e. band edge [103] or stationary inflection point [30]). In general, photonic crystals made up from periodic layers of isotropic materials exhibit 2$\text{nd}$ order band edges. On the other hand, a DBE is associated with a 4$\text{th}$ order $K-\omega$ curve, whereas the MPC medium has a 3$\text{rd}$ order $K-\omega$ curve exhibiting a stationary inflection point (SIP). Chapter 6 demonstrated that partially coupled transmission lines printed on uniform dielectric substrates offer a convenient realization for $K-\omega$ diagrams of degenerate band edge (DBE) crystals. Such coupled line configurations can also be used to realize the magnetic photonic crystal (MPC) dispersion diagrams when a biased ferromagnetic material is used as the substrate [104]. These readily manufacturable and low cost designs are very attractive in developing smaller RF devices including small antennas as in Chapter 6. However, printed DBE and SIP designs usually require many layout iterations using full wave methods (integral equation or finite element approaches). Also, an understanding of various loading effects (i.e. capacitive, inductive) is not as easy
due to special modeling needs of these circuits in the context of rigorous tools. To alleviate these issues, in this chapter, we present a 4-port lumped element circuit for the partially coupled microstrip lines (see Fig. 7.2). Using the transfer matrix approach (see Chapter 2), we demonstrate that the circuit delivers a $K$-$\omega$ dispersion curve identical to the original printed geometry in Fig. 7.1. In this chapter, we also provide a 6-port circuit to model triple coupled lines. This 6-port circuit is shown to exhibit $K$-$\omega$ curve with symmetric SIPs. As such, the 6-port circuit is capable of realizing modes seen in the past only in presence of biased ferromagnetic materials. Thus, the coupled microstrip lines provide a possible approach to realize a new class of modes.

### 7.1 Lumped Element Circuit Model of Printed DBE

Recently, various nonlinear $K$-$\omega$ diagrams were realized using a pair of partially coupled dual lines [51,104] as shown in Fig. 7.1(a). Each line is thought as carrying one of the electric field polarization, say $E_x$ or $E_y$ component. Diagonal anisotropy is realized when different printed line lengths in the uncoupled sections introduce a phase delay. Coupling between the dual lines provides non-diagonal anisotropy. A DBE mode is then realized by appropriately tuning the line thicknesses and coupled line separations [51].

To design the DBE $K$-$\omega$ diagram using the simplest lumped circuit model, we form a two section printed unit cell as in Fig 7.1(b) by combining the uncoupled line sections of the original layout in Fig. 7.1(a). A possible 4-port lumped circuit model of the printed unit cell in Fig. 7.1(b) is depicted in Fig. 7.2(a). Specifically, the uncoupled section of the circuit is modeled using the lumped loads $(L_1, C_1)$ and
Figure 7.1: (a) Partially coupled transmission lines emulating DBE dispersion on a uniform dielectric substrate; (b) Modified version of the partially coupled lines emulating DBE. Uncoupled lines represent a diagonal anisotropy, whereas coupled section creates a general form of (non-diagonal, misaligned) of anisotropy.

Figure 7.2: (a) Lumped circuit model of partially coupled lines in Fig. 7.1(b); (b) Different band edges obtained simply by changing the amount of capacitive coupling $C_M$ in case uncoupled section is capacitively loaded ($L_1 = C_M = L_3 = 1 \text{nH}, C_1 = 10 \text{pF}, C_2 = C_3 = 1 \text{pF},$ and $L_M = 0$).

$L_2, C_2$ per line. For the coupled sections, mutual inductances and/or the capacitor connecting the two circuit branches must be introduced to create a controlled level of coupling for emulation of the non-diagonal anisotropy. Cascade of these uncoupled and coupled sections then forms the equivalent model of the DBE unit cell.

The dispersion diagram of the lumped element DBE crystal is calculated using the well known transfer matrix approach (see Chapter 2 and [31]). For this, first we construct the transfer matrices of the uncoupled ($\overline{T_{UC}}$) and that of the coupled ($\overline{T_C}$)
sections via even - odd mode analysis as described in Appendix D and [92]. Using
the complete transfer matrix \( \mathbf{T} = \mathbf{T}_{UC} \cdot \mathbf{T}_C \), \( \mathbf{v}_{in} = [V_1 \ I_1 \ V_2 \ I_2]^T \) and \( \mathbf{v}_{out} = [V_3 \ I_3 \ V_4 \ I_4]^T \), Bloch’s theorem \( \mathbf{v}_{in} = \mathbf{T}\mathbf{v}_{out} = \mathbf{T}e^{iK}\mathbf{v}_{in} \) is employed to determine the
propagating modes. This is done by solving the eigenvalue equation \( [\mathbf{T} - \mathbf{I}e^{-iK}] \mathbf{v}_{in} = 0 \) where \( K \) denotes the propagation constants plotted in the dispersion diagram of
Fig. 7.2(b).

To demonstrate the equivalence between the lumped element unit cell and the
DBE crystal, we consider a unit cell whose top branch in the uncoupled section is
 capacitively loaded. We also choose \( L_1 = L_2 = L_3 = 1\text{nH} \), \( C_2 = C_3 = 1\text{pF} \), and \( C_1 = 10\text{pF} \). From Fig 7.2(b), it is clear that the dispersion diagram can be significantly
altered by simply varying the coupling capacitor \( C_M \). Specifically, for \( C_M = 1\text{pF} \) the
circuit displays a regular band edge (RBE). However, for \( C_M = 2.3\text{pF} \) the resulting \( K-\omega \) curve displays a DBE and if \( C_M = 4\text{pF} \), a double band edge (DbBE) is obtained [29].

An important use of the lumped circuit model is to provide guidelines for designing
DBE \( K-\omega \) diagram. For example in above case, changing the value of the mutual
inductance \( (L_M) \) value (for constant \( C_M) \) does not affect the dispersion diagram.
However, when the uncoupled branch is inductively loaded (i.e. \( L_1 = 10\text{nH}, C_1 = 1\text{pF} \) the same DBE behavior is obtained for \( L_M = 2.3\text{nH} \). That is, an appropriate
coupling mechanism (depending on the loading type) must be present to realize the
DBE. This circuit model also allows for a convenient approach to develop reduced
size printed DBE antennas presented in Chapter 6 and [53].
7.2 Printed DBE Design Using Dual Transmission Lines Loaded with Lumped Circuit Elements

7.2.1 DBE Circuit Example

To demonstrate the advantage of the lumped circuit model, we proceed with the design of a new DBE circuit that will miniaturize the printed line layout depicted in Fig. 7.1(a). As can be predicted, thinner printed lines are highly desirable in the layout to reduce the electrical size of a DBE unit cell. However, thin transmission lines exhibit high inductance per unit length values and therefore weaken the coupling capacitor effect on the dispersion diagram. On the other hand, the $K-\omega$ bending via coupling capacitor is maximized when the transmission lines are capacitively loaded. Unfortunately, this necessitates the use of thick transmission lines that result in an electrically large unit cell. Hence, the proposed layout in Fig. 7.3(a) uses a combination of thick and thin transmission lines to achieve the capacitively tuned DBE dispersion with concurrent electrical size reduction. The layout is on a 100mil thick Rogers Duroid substrate having a relative dielectric constant of $\epsilon_r = 2.2$ and loss factor of $\tan \delta = 0.0009$. The total inductance and capacitance of the printed lines are approximated from the per unit length impedance values obtained through microwave circuit simulators (such as Advanced Design Systems, Ansoft Designer, or Ansoft Serenade). Specifically, the extracted circuit parameters are $L_1 = 3.26\text{nH}, C_1 = 2.66\text{pF}, L_2 = 11.04\text{nH}, C_2 = 0.70\text{pF}, L_3 = 5.52\text{nH},$ and $C_3 = 0.35\text{pF}$. Tuning the coupling capacitor to $C_M = 1.0\text{pF}$ results in a DBE at $2.1\text{GHz}$ as depicted by the blue curve of Fig. 7.3(b). When the $K-\omega$ diagram of the DBE circuit is computed using a full wave electromagnetic solver (Ansoft HFSSv10, see the red curve in Fig. 7.3(b)), we observe a very good agreement with the lumped element circuit model.
Figure 7.3: (a) Transmission line implementation of lumped DBE circuit on 100mil thick Duroid substrate; (b) Dispersion diagram of the printed DBE cell on the left. Equivalent circuit model analysis agrees well with the full wave simulation.

However, due to the additional edge couplings accounted only in the full wave model, the coupling capacitor used in the numerical computation $C_M = 0.5 \text{pF}$ is smaller than that of the equivalent circuit model. We note that the new DBE layout is 25% smaller as compared to the transmission line implementation in Fig. 7.1(a) without the lumped capacitor. Obviously, future replacement of the thick transmission line with a lumped shunt capacitor is expected to lead to further size miniaturization.

### 7.2.2 DBE Antenna Example

As demonstrated in Chapter 6, the maximally flat 4th order DBE band diagram pushes the $K = \pi$ points further down in frequency implying miniaturization of resonator geometry. Since antennas composed of 2 unit cells initially resonate at $K = \pi$ frequencies (such as a loop antenna where half of the loop is thought as the unit cell), printed DBE unit cells offer additional size reduction as compared to traditional loadings. To harness the maximally flat DBE band diagram and realize a reduced size antenna we propose the lumped DBE antenna circuit in Fig. 7.4(a) with circular periodicity. The coupling between the two circuit branches is achieved
Figure 7.4: (a) Lumped element model of DBE antenna composed of circularly cascaded 2 unit cells; (b) Lumped capacitor loaded dual transmission lines implementing DBE unit cell on 125mil thick Duroid substrate and their corresponding dispersion diagram. $K = \pi$ resonances are taken to lower frequencies by switching to a DBE $K-\omega$ curve.

via a lumped capacitor, whereas one of the uncoupled line branches is capacitively loaded ($C_L = 1\,\text{pF}$). Fig. 7.4(b) demonstrates the loaded dual transmission line layout implementing the DBE circuit on a 125mil thick Duroid substrate. As expected from the design guidelines, the bending of the circuit’s $K-\omega$ curve and the location of the $K = \pi$ resonance is easily controlled by changing the coupling amount. Specifically, the circuit exhibits a DBE dispersion around 2.25GHz for $C_M = 0.65\,\text{pF}$.

The geometrical layout and performance of the DBE antenna is depicted in Fig. 7.5(a) and (b), respectively. The antenna is printed on a $2'' \times 2''$, 125mil thick Duroid substrate. As usual, the back side of the substrate is used as the ground plane. For impedance matching, the antenna is fed with a capacitively coupled 50Ω coaxial cable. This feeding mechanism provides an excellent match (i.e. $|S_{11}| <$
Figure 7.5: (a) DBE antenna layout on a 2" × 2", 125mil thick Duroid substrate; (b) $|S_{11}| < -10$dB resonances and bandwidths computed for different coupling capacitors.

$-20$dB) at the DBE frequency of 2.25GHz (see Fig. 7.5(b). The resonance of the antenna is lowered using larger coupling capacitances. In all of the three cases, the antenna has 6.9dB broadside gain. On the other hand, this design suffers from smaller bandwidths as the resonance frequency is lowered. Specifically, the design has a bandwidth of 0.9% for $C_M = 0.0pF$, 0.4% for $C_M = 0.5pF$, and 0.25% for $C_M = 0.65pF$. Nevertheless, the substrate thickness can be adjusted to compensate the bandwidth without compromising other antenna parameters. For instance, for the same antenna footprint increasing the substrate thickness to 375mil results in a considerably larger bandwidth of 1.5% for $C_M = 0.65pF$.

To verify the antenna design, we proceeded to manufacture it on a 2" × 2", 125mil thick Duroid substrate as shown in Fig. 7.6(a). The dual transmission lines of the antenna were loaded with S series R05 high Q capacitors obtained from Johanson Technology, CA, USA (www.johansontechnology.com). As seen from the measured $|S_{11}|$ curves in Fig. 7.6(b), a coupling capacitor of $C_M = 0.5pF$ shifts the antenna resonance from 2.85GHz to 2.35GHz and results in a 0.4% bandwidth. On the other hand, the measured directivity shown in Fig. 7.6(a) agrees well with the computed
6.9dB directivity. However, due to the high losses associated with the chip capacitors, the realized gain is only about 1.8dB as compared to 6.3dB gain of the unloaded $C_M = 0\,\text{pF}$ antenna. We expect that this rather low 35% antenna efficiency can be substantially improved using interdigital microstrip capacitors instead of lumped elements.

### 7.3 Triple Coupled Lines for SIP Realization

Following the same logic as above, we may present a different circuit model to emulate other dispersion diagrams such as a 3\textsuperscript{rd} order $K\omega$ curve exhibiting a stationary inflexion point (SIP). So far, SIPs have only been realized by modifying the DBE $K\omega$ curves with biased ferrites (see Chapter 2). In these configurations, the non-reciprocal behavior of the ferrite material is used to create spectral asymmetry, i.e. $\omega(K) \neq \omega(-K)$ to generate a 3\textsuperscript{rd} order $K\omega$ behavior around a certain frequency. Therefore, realization of SIP on uniform substrates will provide a much simpler unit
cell structure that could then provide experimental verification of the frozen mode regime.

Since printed lines on uniform substrates are reciprocal, at least a 6\textsuperscript{th} order K-ω behavior is necessary to observe the 3\textsuperscript{rd} order K-ω curve exhibiting the SIP. To do so, we introduce the lumped unit cell shown in Fig. 7.7(a) to model three partially coupled printed lines. Due to the capacitively loaded uncoupled branches (C\textsubscript{11} = 10pF, C\textsubscript{12} = 5pF, see Fig. 7.7(a) for other circuit parameters), a capacitive coupling mechanism can be used to create mode degeneracy. The adjacent circuit branches are equally coupled to each other (C\textsubscript{M12} = C\textsubscript{M23} = 2pF), whereas the coupling C\textsubscript{M13} between the top and bottom branches is used to tune the dispersion curve. To calculate the dispersion diagram, the eigenvalue equation was extended to handle the 6-port transfer matrices (see Appendix D) of the uncoupled and coupled sections. As expected, a 6\textsuperscript{th} order K-ω behavior is indeed observed when C\textsubscript{M13} is comparable to the other coupling capacitors (see Fig. 7.7(b)). From Fig. 7.7(c) we also observe that the SIP resonance can be tuned simply by varying C\textsubscript{M13}. Specifically, for C\textsubscript{M13} = 2.6pF, an SIP resonance occurs at 2.605GHz. Further, it is possible to realize very slow positive group velocities (C\textsubscript{M13} = 2.7pF, \(\partial \omega / \partial K > 0\)) or very slow negative group velocities (C\textsubscript{M13} = 2.5pF, \(\partial \omega / \partial K < 0\)) by adjusting the amount of coupling between the top and bottom branches.

7.4 Remarks

This chapter presented multi-port lumped circuit models to emulate propagation modes found in natural multilayered and possibly anisotropic media. Coupled dual microstrip lines were shown to exhibit a K-ω curve supporting a DBE. In addition,
Figure 7.7: (a) Equivalent circuit model of a partially coupled triple transmission lines; mutual coupling is achieved via the capacitors $C_{M12}$, $C_{M23}$, and $C_{M13}$; (b) 6th order K-ω curve exhibiting symmetric stationary inflection points; $L_{ij} = 1\text{nH}$, $C_{11} = 10\text{pF}$, $C_{12} = 5\text{pF}$, $C_{13} = 1\text{pF}$, $C_{2j} = 1\text{pF}$, $C_{M12} = C_{M23} = 2\text{pF}$ ($i=1, 2; j = 1, 2, 3$). (c) Coupling between first and third lines ($C_{M13}$) controls the K-ω slope around the SIP frequency.

Triple coupled transmission lines were shown to realize a K-ω curve with symmetric SIP resonances. The simplicity of the lumped element models and their demonstrated ability to realize different dispersion diagrams is promising for new RF applications including antenna and circuit miniaturization. Specifically, we demonstrated DBE circuit and antenna examples built from dual printed lines loaded with lumped capacitors. Both examples achieved 20-25% size reduction, whereas the DBE antenna
efficiency (35%) suffered from lumped element losses. Future work will concentrate on high contrast substrate implementations and interdigital capacitor loads in order to enhance miniaturization and gain levels of the DBE antennas.
CHAPTER 8

CONCLUSIONS

Engineered materials have been the focus of extensive research in electromagnetics as they offer new possibilities for novel, smaller, light-weight, and multifunctional RF devices. In particular, periodic structures that exploit new modal phenomenology for extraordinary and unique propagation properties have attracted considerable interest. This dissertation investigates a recent metamaterial concept based on periodic assemblies of layered anisotropic media and possibly magnetic materials. These assemblies, so called the magnetic photonic (MPC) and degenerate band edge (DBE) crystals, display unique higher order band diagrams and therefore offer new resonant modes and degrees of freedom to optimize performance of antennas and RF applications.

Earlier studies of MPC and DBE crystals have primarily focused on semi-infinite settings made up from idealistic material parameters. This dissertation is the first to investigate the fundamental concepts of realistic MPC/DBE assemblies and use them to realize novel conformal antennas. Our new contributions can be summarized in two groups:

A) Bulk Crystals

1. Characterization of wave propagation within 1D MPC/DBE crystals using real material parameters,
2. Demonstration of directive radiation from miniature antennas embedded within 1D realistic MPC/DBE crystals,

3. Development of computational tools and measurement methods to design and fabricate high gain DBE antennas,

B) Printed Circuits

1. Emulation of DBE properties on uniform microwave substrates via partially coupled transmission lines,

2. Introduction of a new class of printed resonator antennas miniaturized using the DBE properties,

3. Development of lumped element circuit models for anisotropic DBE/MPC crystals leading to convenient design guidelines and smaller antennas by combining printed lines and lumped circuit elements.

To summarize, Chapter 2 demonstrated that MPCs constructed from periodic arrangements of available material layers can indeed support unidirectional propagation properties. These frozen modes exhibit dramatic wave slow-down coupled with large field amplitudes even the material layers have finite loss tangents. These phenomena occur at or near a specific frequency referred to as the stationary inflection point (SIP). We demonstrated that SIP and the associated frozen mode offers improved antenna gain, matching, and miniaturization.

Chapter 3 considered radiation performances of small dipoles embedded within finite thick 1D MPC and DBE crystals using a spectral domain method of moment (MoM) formulation. It was shown that miniature dipoles within MPCs and DBE
crystals exhibit directive radiation due to slow mode resonances associated with the 3rd and 4th order MPC and DBE dispersion curves, respectively. Moreover, material losses were shown to be acceptable provided that the crystal layers have loss factors smaller than $10^{-4}$.

In Chapter 4, a full-wave MoM solution of the surface integral equation (SIE) was carried out to model antennas embedded within 3D DBE crystals made up from high contrast uniaxial dielectric materials. The SIE-MoM employed the closed form dyadic Green’s functions (GF) of uniaxial dielectrics. To our knowledge, this is the first time that uniaxial GFs are used within SIE-MoM context. A significant advantage in MoM matrix filling time is obtained in contrast to the numerically costly spectral domain uniaxial GFs. It was also demonstrated that 3D finite DBE crystals can achieve directive radiation (similar to 1D case) when the lateral dimensions are carefully designed.

Chapter 5 presented a measurement setup to characterize low loss engineered or natural uniaxial dielectric materials. Using this rectangular prism–shaped cavity setup, effective parameters for textured uniaxial dielectrics were characterized for the first time. After detecting the most suitable adhesive choice for low loss textured uniaxial material fabrication, a 3D DBE antenna prototype was built. It was shown that the volumetric DBE antenna performed with a large radiation efficiency and offered an almost optimal (100%) directivity with respect to its aperture size.

In Chapter 6, a novel concept of partially coupled dual transmission lines emulating DBE propagation properties on uniform microwave substrates was introduced. This dual printed line emulation enables straightforward and low cost realization of novel RF devices that can utilize the DBE properties. Specifically, the maximally
flat DBE dispersion diagram was harnessed, for the first time, to design and realize miniature printed antennas. A fabricated miniature DBE antenna was shown to perform almost at the fundamental gain × bandwidth product limit. This antenna had a $\lambda_0/9 \times \lambda_0/9$ footprint with 3% bandwidth and 4.5dB realized gain. In addition, a high gain dual-polarized DBE antenna array configuration built on a low contrast substrate was also proposed. This $4 \times 4$ antenna array had $1.2\lambda_0 \times 1.2\lambda_0$ aperture size; achieved 111% aperture and 75% radiation efficiencies.

Finally, Chapter 7 developed multi-port lumped circuit models to emulate propagation properties of DBE and MPC crystals. A dual line coupling mechanism was shown to exhibit a 4th order dispersion diagram with a DBE mode, whereas triple coupling was used to realize a 6th order dispersion with symmetric SIP resonances. The simplicity of the lumped element model was used to derive better design guidelines to build miniature DBE antennas employing lumped or interdigital capacitors.

This dissertation clearly demonstrated that the unique propagation characteristics of MPCs and DBE crystals can be used for antenna and array enhancements. Although an experimental demonstration of a volumetric DBE antenna was presented, computational and fabrication challenges associated with high gain DBE/MPC assemblies are still under investigation. Additionally, DBE design guidelines obtained from the lumped circuit model must further be studied to address bandwidth and loss issues associated with capacitive loadings. In this context, future work will concentrate on high contrast substrate implementations and interdigital capacitor loads to enhance miniaturization and gain levels of the printed DBE antennas.
APPENDIX A

TRANSFER MATRIX OF A 1D HOMOGENOUS LAYER WITH ANISOTROPIC MATERIAL PARAMETERS

Homogenous, non-dispersive, linear, and anisotropic material regions can be represented via the usual constitutive relations between \((E, D)\) and \((B, H)\) with fully populated permittivity \(\varepsilon_r\) and permeability \(\mu_r\) tensors. In such source free material regions, Maxwell’s equations take the form

\[
\nabla \times E(r) = \frac{i\omega}{c} \mu_r \cdot H(r), \tag{A.1a}
\]

\[
\nabla \times H(r) = -\frac{i\omega}{c} \varepsilon_r \cdot E(r). \tag{A.1b}
\]

Plane wave solutions to the above equations satisfy the relations

\[
E(r) = e^{i(k_xx x + k_y y)} E(z), \tag{A.2a}
\]

\[
H(r) = e^{i(k_xx x + k_y y)} H(z). \tag{A.2b}
\]

Substituting the plane wave solutions and writing the \(\nabla\) operator explicitly in (A.1a) and (A.1b), we get

\[
\begin{bmatrix}
\frac{\partial}{\partial x} \hat{x} + \frac{\partial}{\partial y} \hat{y} + \frac{\partial}{\partial z} \hat{z} \end{bmatrix} \times \begin{bmatrix} E_x(z) \\ E_y(z) \\ E_z(z) \end{bmatrix} e^{i(k_xx x + k_y y)} = \frac{i\omega}{c} \begin{bmatrix} \mu_{xx} & \mu_{xy} & \mu_{xz} \\ \mu_{yx} & \mu_{yy} & \mu_{yz} \\ \mu_{zx} & \mu_{zy} & \mu_{zz} \end{bmatrix} \cdot \begin{bmatrix} H_x(z) \\ H_y(z) \\ H_z(z) \end{bmatrix} e^{i(k_xx x + k_y y)}, \tag{A.3a}
\]

\[
\begin{bmatrix}
\frac{\partial}{\partial x} \hat{x} + \frac{\partial}{\partial y} \hat{y} + \frac{\partial}{\partial z} \hat{z} \end{bmatrix} \times \begin{bmatrix} H_x(z) \\ H_y(z) \\ H_z(z) \end{bmatrix} e^{i(k_xx x + k_y y)} = -\frac{i\omega}{c} \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix} \cdot \begin{bmatrix} E_x(z) \\ E_y(z) \\ E_z(z) \end{bmatrix} e^{i(k_xx x + k_y y)}. \tag{A.3b}
\]
Next, performing the derivative operations in the above and cancelling the exponents, we obtain

\[
\begin{bmatrix}
  ik_y E_z(z) - \frac{\partial}{\partial z} E_y(z) \\
  -ik_x E_z(z) + \frac{\partial}{\partial z} E_x(z) \\
  ik_x E_y(z) - ik_y E_z(z)
\end{bmatrix}
= \frac{i\omega}{c} \begin{bmatrix}
  \mu_{yx} H_x(z) + \mu_{xy} H_y(z) + \mu_{xz} H_z(z) \\
  \mu_{yz} H_x(z) + \mu_{zy} H_y(z) + \mu_{zz} H_z(z) \\
  \mu_{zx} H_x(z) + \mu_{zy} H_y(z) + \mu_{zz} H_z(z)
\end{bmatrix}
\]  

(A.4a)

\[
\begin{bmatrix}
  ik_y H_z(z) - \frac{\partial}{\partial z} H_y(z) \\
  -ik_x H_z(z) + \frac{\partial}{\partial z} H_x(z) \\
  ik_x H_y(z) - ik_y H_z(z)
\end{bmatrix}
= -\frac{i\omega}{c} \begin{bmatrix}
  \epsilon_{yx} E_x(z) + \epsilon_{xy} E_y(z) + \epsilon_{xz} E_z(z) \\
  \epsilon_{yz} E_x(z) + \epsilon_{zy} E_y(z) + \epsilon_{zz} E_z(z) \\
  \epsilon_{zx} E_x(z) + \epsilon_{zy} E_y(z) + \epsilon_{zz} E_z(z)
\end{bmatrix}
\]  

(A.4b)

From the last row of (A.4a) and (A.4b), the z components of the fields can be related to the x and y components as

\[
H_z(z) = [n_x E_y(z) - n_y E_x(z) - \mu_{zx} H_x(z) - \mu_{zy} H_y(z)] \mu_{zz}^{-1}, \quad (A.5a)
\]

\[
E_z(z) = [-n_x H_y(z) + n_y H_x(z) - \epsilon_{zx} E_x(z) - \epsilon_{zy} E_y(z)] \epsilon_{zz}^{-1}, \quad (A.5b)
\]

where

\[
n_x = \frac{k_x c}{\omega} = \frac{k_x}{k_0}, \quad n_y = \frac{k_y c}{\omega} = \frac{k_y}{k_0}, \quad (A.6)
\]

Since \(E_z(z)\) and \(H_z(z)\) can be directly obtained from (A.5a) and (A.5b), it is not necessary to carry them in further equations. Defining

\[
\Psi(z) = \begin{bmatrix} E_x(z) \\ E_y(z) \\ H_x(z) \\ H_y(z) \end{bmatrix}
\]  

(A.7)

as the plane wave mode vector, (A.4a) and (A.4b) can be recast into

\[
\frac{\partial}{\partial z} \Psi(z) = \frac{i\omega}{c} \overrightarrow{M} \cdot \Psi(z), \quad (A.8)
\]

where \(\overrightarrow{M}\) is given by

\[
\overrightarrow{M} = \begin{bmatrix} \overrightarrow{M}_{11} & \overrightarrow{M}_{12} \\ \overrightarrow{M}_{21} & \overrightarrow{M}_{22} \end{bmatrix}
\]  

(A.9)
with

\[
\overline{M}_{11} = \begin{bmatrix}
-\frac{\epsilon_{zx}}{\epsilon_{zz}} n_x - \frac{\mu_{yz}}{\mu_{zz}} n_y & \left(\frac{-\epsilon_{zy}}{\epsilon_{zz}} + \frac{\mu_{yz}}{\mu_{zz}}\right) n_x \\
-\frac{\epsilon_{zx}}{\epsilon_{zz}} n_y - \frac{\mu_{zy}}{\mu_{zz}} n_x & \left(\frac{-\epsilon_{zy}}{\epsilon_{zz}} + \frac{\mu_{zy}}{\mu_{zz}}\right) n_y
\end{bmatrix}, \quad (A.10a)
\]

\[
\overline{M}_{12} = \begin{bmatrix}
\mu_{yx} - \frac{\mu_{zx} \mu_{yz}}{\mu_{zz}} + \frac{n_x n_y}{\epsilon_{zz}} & \mu_{yy} - \frac{\mu_{yz} \mu_{zy}}{\mu_{zz}} - \frac{n_y^2}{\epsilon_{zz}} \\
-\mu_{xx} + \frac{\mu_{zx} \mu_{xx}}{\mu_{zz}} + \frac{n_x^2}{\epsilon_{zz}} & -\mu_{xy} + \frac{\mu_{yz} \mu_{zy}}{\mu_{zz}} - \frac{n_x n_y}{\epsilon_{zz}}
\end{bmatrix}, \quad (A.10b)
\]

\[
\overline{M}_{21} = \begin{bmatrix}
-\epsilon_{yx} - \frac{\epsilon_{zx} \epsilon_{yz}}{\epsilon_{zz}} - \frac{n_x n_y}{\mu_{zz}} & \epsilon_{yy} - \frac{\epsilon_{zy} \epsilon_{zy}}{\epsilon_{zz}} + \frac{n_y^2}{\epsilon_{zz}} \\
\epsilon_{xx} - \frac{\epsilon_{zx} \epsilon_{xx}}{\epsilon_{zz}} - \frac{n_x^2}{\mu_{zz}} & \epsilon_{xy} - \frac{\epsilon_{zy} \epsilon_{zy}}{\epsilon_{zz}} + \frac{n_x n_y}{\mu_{zz}}
\end{bmatrix}, \quad (A.10c)
\]

\[
\overline{M}_{22} = \begin{bmatrix}
-\epsilon_{yz} n_x - \frac{\mu_{zx}}{\mu_{zz}} n_y & \left(\frac{\epsilon_{yz}}{\epsilon_{zz}} - \frac{\mu_{zy}}{\mu_{zz}}\right) n_x \\
\left(\frac{\epsilon_{zx}}{\epsilon_{zz}} - \frac{\mu_{zx}}{\mu_{zz}}\right) n_y & -\epsilon_{yz} n_x - \frac{\mu_{zy}}{\mu_{zz}} n_y
\end{bmatrix}. \quad (A.10d)
\]

We next proceed by introducing the transfer matrix \( \overline{T} \) which relates fields at \( z = 0 \) to those at \( z \) as

\[
\Psi(z) = \overline{T}(z) \cdot \Psi(0). \quad (A.11)
\]

Substituting the above definition in (A.8) results in

\[
\frac{\partial}{\partial z} \overline{T}(z) = \frac{i \omega}{c} \overline{M} \cdot \overline{T}(z). \quad (A.12)
\]

Solution of the above equation can be written as

\[
\overline{T}(z) = e^{i \frac{\omega}{c} \overline{M} z} = \overline{S} \cdot e^{i \frac{\omega}{c} \overline{X} z} \cdot \overline{S}^{-1}, \quad (A.13)
\]

where \( \overline{S} \) and diagonal \( \overline{X} \) are the matrices consisting of eigenvectors and eigenvalues of \( \overline{M} \), respectively. \( \overline{S} \) and \( \overline{X} \) satisfies the relation

\[
\overline{M} = \overline{S} \cdot \overline{X} \cdot \overline{S}^{-1}. \quad (A.14)
\]
From the transfer matrix expression in A.13, we observe that field modes and corresponding wavenumbers of the medium are associated with the eigenvector-eigenvalue pairs of $\overline{M}$. 
Fields radiated by a Hertzian electric current $\mathbf{J} = \hat{\mathbf{j}}\delta(x)\delta(y)\delta(z)$ located in an unbounded homogenous medium can be determined from the solution of Helmholtz wave equation. These fields are given by (an $e^{-i\omega t}$ time convention is assumed and suppressed)

$$
\mathbf{E}(\mathbf{r}) = \frac{i\eta_0}{k} \int_{V'} \left[ k^2 \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right] G(R) \cdot \mathbf{J}(\mathbf{r}') \, dx'dy'dz',
$$

$$
\mathbf{H}(\mathbf{r}) = \int_{V'} \left[ \begin{array}{ccc}
\frac{\partial}{\partial x} & 0 & -\frac{\partial}{\partial y} \\
0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
-\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0
\end{array} \right] G(R) \cdot \mathbf{J}(\mathbf{r}') \, dx'dy'dz',
$$

where $k = \omega\sqrt{\epsilon\mu}$ is the wavenumber; $\omega$ is the angular frequency; $\epsilon$ and $\mu$ are the permittivity and permeability of the material, respectively. In the above, $R = |\mathbf{R}| = |\mathbf{r} - \mathbf{r}'|$ is the distance from the source location $\mathbf{r}'$ to the observation point $\mathbf{r}$, and $G(R) = e^{ikR}/(4\pi R)$ is the scalar Green’s function which satisfies the equation

$$(\nabla^2 + k^2)G(R) = -\delta(\mathbf{r} - \mathbf{r}').$$

To determine the spectral domain Green’s function, we start by introducing the Fourier Transform pairs.
\[ \tilde{f}(k_x, k_y, k_z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y, z) e^{-i(k_x x + k_y y + k_z z)} dx dy dz \]

\[ f(x, y, z) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{f}(k_x, k_y, k_z) e^{i(k_x x + k_y y + k_z z)} dk_x dk_y dk_z. \] (B.3)

Subsequently, we use the convolution property

\[ f(x, y, z) * g(x, y, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y, z) g(x', y', z') dx' dy' dz' \] (B.4)

\[ = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{f} \tilde{g} e^{i(k_x x + k_y y + k_z z)} dk_x dk_y dk_z, \] (B.5)

in (B.1) to rearrange it as

\[ E(r) = \frac{i\eta_0}{(2\pi)^3 k} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \begin{array}{ccc} k^2 + \frac{\partial^2}{\partial x^2} & \frac{\partial^2}{\partial x \partial y} & \frac{\partial^2}{\partial x \partial z} \\ \frac{\partial^2}{\partial y \partial x} & k^2 + \frac{\partial^2}{\partial y^2} & \frac{\partial^2}{\partial y \partial z} \\ \frac{\partial^2}{\partial z \partial x} & \frac{\partial^2}{\partial z \partial y} & k^2 + \frac{\partial^2}{\partial z^2} \end{array} \right] \tilde{G} \cdot \tilde{J} dk_x dk_y dk_z \]

\[ H(r) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \begin{array}{ccc} 0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{array} \right] \tilde{G} \cdot \tilde{J} dk_x dk_y dk_z, \] (B.6)

where \( \tilde{J} = \hat{j} \). Spectral form of the scalar Green’s function \( \tilde{G} \) of the above can be derived by applying Fourier transform to both sides of (B.2). Hence, \( \tilde{G} \) is explicitly given by

\[ \tilde{G}(k_x, k_y, k_z) = \frac{1}{k_x^2 + k_y^2 + k_z^2 - k^2}. \] (B.7)

The triple integrations of (B.6) can be decreased by an order when the integral along \( k_z \) is analytically evaluated. However, for lossless materials (i.e. \( k \) real) the integrand exhibits pole singularities at \( k_z = \pm \sqrt{k^2 - k_x^2 - k_y^2} \). To alleviate this issue, a small amount of loss is assumed within the material and the poles are moved away from the real axis. Following this, the integration contour is closed either from \(+i\infty\) or \(-i\infty\) to satisfy the radiation condition for the chosen sign of \( z \). With the radiation
condition satisfied, the modified contour leads to the same value with the original path according to Jordan’s lemma. Applying Cauchy’s integration formula for the closed integration contour then leads to [34]

\[
\frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{k_x^2 + k_y^2 + k_z^2 - k^2} e^{i(k_x x + k_y y + k_z z)} dk_x dk_y dk_z = \frac{i}{8\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{i(k_x x + k_y y + k_z z)}}{k_z} dk_x dk_y.
\]

in which \( k_z = \sqrt{k^2 - k_x^2 - k_y^2} \). (B.8)

Finally, substituting (B.8) into (B.6) and carrying out the derivative operations (i.e. \( \frac{\partial}{\partial x} \rightarrow ik_x \), \( \frac{\partial}{\partial y} \rightarrow ik_y \), and \( \frac{\partial}{\partial z} \rightarrow ik_z \)) results in

\[
E(r) = -\frac{\eta_0}{8\pi^2 k} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \begin{bmatrix} k^2 - k_x^2 & -k_x k_y & \mp k_x k_z \\ -k_x k_y & k^2 - k_y^2 & \mp k_y k_z \\ \mp k_x k_z & \mp k_y k_z & k^2 - k_z^2 \end{bmatrix} e^{ik_z \mp z} \hat{j} e^{i(k_x x + k_y y)} dk_x dk_y
\]

\[
H(r) = -\frac{1}{8\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \begin{bmatrix} 0 & \mp k_z & k_y \\ \pm k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix} e^{ik_z \pm z} \hat{j} e^{i(k_x x + k_y y)} dk_x dk_y.
\] (B.9)

The above representation can further be generalized to any source location (i.e. \( J = \hat{j}\delta(x - x')\delta(y - y')\delta(z - z') \)) simply by using the coordinate transforms \( x \to x - x' \), \( y \to y - y' \), and \( z \to z - z' \).
APPENDIX C

FINITE ELEMENT METHOD FOR DETERMINING CAVITY RESONANCES

Throughout Chapter 5, we used FEM to solve for the eigenvalues of (5.4) for the cavity–sample system. FEM is well justified due to its flexibility in modeling arbitrary textures forming the composition of the engineered materials. For our case we used FEM to discretize the cavity via a connected volumetric mesh of hexahedral elements. The numerical Rayleigh-Ritz procedure was adapted to generate the matrix system. To do so, we start with the wave equation (5.4) in a source free region and introduce the discrete expansion

\[ E(r) = \sum_{i=1}^{N} x_i e_i(r), \]  

where \( x_i \) are the unknown coefficients of the expansion and \( e_i(r) \) are the tangential edge-based vector basis functions defined on curvilinear hexahedral finite elements (see \([48, 65]\) for a detailed discussion of curvilinear finite element method). Hence, (5.4) can be rewritten as

\[ \sum_{i=1}^{N} x_i \left[ \nabla \times \nabla \times e_i(r) - k_0^2 \varepsilon_r \cdot e_i(r) \right] = 0 \]  

Next on, applying Galerkin’s testing \([48]\) results in

\[ \int_{v_j} d\varepsilon_j e_i(r) \cdot \left\{ \sum_{i=1}^{N} x_i \left[ \nabla \times \nabla \times e_i(r) - k_0^2 \varepsilon_r \cdot e_i(r) \right] \right\} = 0 \]  

121
for \( j = 1, \ldots, N \). This is further simplified to

\[
\sum_{i=1}^{N} x_i \left\{ \int_{v_j} dv \nabla \times e_j(r) \cdot \nabla \times e_i(r) - k_0^2 \int_{v_j} dv e_j(r) \cdot \bar{\epsilon}_r \cdot e_i(r) \right\} = 0 \tag{C.4}
\]

using the properties of the basis functions in (C.1) and the boundary conditions on the cavity walls. This expression can now be cast into a generalized eigenvalue equation

\[
[A] \{x\} - \lambda_{\text{eig}} [B] \{x\} = 0 \tag{C.5}
\]

where

\[
A_{ji} = \int_{v_j} dv \left[ \nabla \times e_j(r) \cdot \nabla \times e_i(r) \right], \quad B_{ji} = \int_{v_j} dv \left[ e_j(r) \cdot \bar{\epsilon}_r \cdot e_i(r) \right],
\]

and, \( \lambda_{\text{eig}} = k_0^2 \) \tag{C.6}

are the eigenvalues. The solution of (C.6) then provides the resonant frequencies \( \omega = \sqrt{\lambda_{\text{eig}}} c \) of the loaded cavity.

As already mentioned, this hexahedral finite element modeling is particularly suited for rectangular cavities and the rectangular sample geometries. Of importance is that the anisotropic nature of the sample under test can be readily incorporated into (C.6). Upon solving (C.5), the resulting fields can be substituted in (5.5) to determine the filling ratios. Also, the geometrical factor \( G \) in (5.3) can be computed from the eigensolutions (C.5) using

\[
H \cdot H^* = (\nabla \times E) \cdot (\nabla \times E)^*, \tag{C.7}
\]

which is readily available in the elements comprising \( [A]_{ij} \).
APPENDIX D

TRANSFER MATRICES OF LUMPED ELEMENT
DBE/MPC CIRCUIT MODELS

D.1 2-port Transmission Line Circuit Model

To derive the transfer matrices of the uncoupled and coupled sections of the lumped element 4-6 port DBE/MPC circuit models, let us first consider the conventional 2-port transmission line model shown in Fig. D.1. \( Z_1 = -i\omega L_R \) and \( Z_2 = -1/i\omega C_R \) denote the impedances of a series inductor \( L_R \) and a shunt capacitor \( C_R \), respectively (an \( e^{-i\omega t} \) time convention is assumed and suppressed). Since transfer matrix \( \mathbf{T}_{RH} \) satisfies the relation

\[
\begin{bmatrix}
V_1 \\
I_1
\end{bmatrix} =
\begin{bmatrix}
T_{11} & T_{12} \\
T_{21} & T_{22}
\end{bmatrix}
\begin{bmatrix}
V_2 \\
I_2
\end{bmatrix},
\]

(D.1)

the individual elements of \( \mathbf{T}_{RH} \) can be obtained by considering open and short circuit terminations of the output port. When the output is terminated with an open circuit implying \( I_2 = 0 \), (D.1) readily gives \( T_{11} = V_1/V_2 \) and \( T_{21} = I_1/V_2 \). Applying Kirchoff’s voltage and current laws in the equivalent circuit then results in

\[
T_{11} = (Z_1 + Z_2)/Z_2 = 1 + Z_1/Z_2, \quad \text{and} \quad (D.2a)
\]

\[
T_{21} = 1/Z_2. \quad (D.2b)
\]
Figure D.1: General lumped element circuit model for a conventional transmission line. \( Z_1 \) and \( Z_2 \) are the impedances of series inductor \( L_R \) and shunt capacitor \( C_R \), respectively.

On the other hand, short circuit termination of the output port implies \( V_2 = 0 \), and (D.1) leads to \( T_{12} = V_1/I_2 \) and \( T_{22} = I_1/I_2 \). Since \( I_2 = I_1 \) for the short termination, the remaining elements of \( \mathbf{T}_{RH} \) become

\[
T_{12} = Z_1, \quad \text{and} \\
T_{21} = 1. \quad \text{(D.3a)}
\]

Substituting (D.2) and (D.3) in (D.1) leads to

\[
\mathbf{T}_{RH} = \begin{bmatrix}
1 + Z_1/Z_2 & Z_1 \\
1/Z_2 & 1
\end{bmatrix} = \begin{bmatrix}
1 - \omega^2 L_R C_R & -i \omega L_R \\
-i \omega C_R & 1
\end{bmatrix}. \quad \text{(D.4)}
\]

D.2 Uncoupled Dual Transmission Line Circuit Model

Fig. D.2 demonstrates the equivalent circuit model of dual uncoupled transmission lines within the printed DBE layout. The transfer matrix \( \mathbf{T}_{UC} \) is defined similar to (D.1), however in this case it relates two input ports to two output ports as (see Fig. D.2 for port numbering)

\[
\begin{bmatrix}
V_1 \\
I_1 \\
V_2 \\
I_2
\end{bmatrix} = \begin{bmatrix}
T_{11} & T_{12} & T_{13} & T_{14} \\
T_{21} & T_{22} & T_{23} & T_{24} \\
T_{31} & T_{32} & T_{33} & T_{34} \\
T_{41} & T_{42} & T_{43} & T_{44}
\end{bmatrix} \begin{bmatrix}
V_3 \\
I_3 \\
V_4 \\
I_4
\end{bmatrix}. \quad \text{(D.5)}
\]
Figure D.2: Equivalent lumped circuit model for the uncoupled line sections of the printed DBE circuit. Different loadings on separate branches emulate the length difference of the uncoupled lines.

Since ports 1-3 and 2-4 are only connected to each other; \( \mathbf{T}_{UC} \) can readily be derived by using the transfer matrix of the regular printed line in (D.4). Noting that the matrix elements relating port 1 to 4 and 2 to 3 are zero, \( \mathbf{T}_{UC} \) can be written as

\[
\mathbf{T}_{UC} = \begin{bmatrix}
1 - \omega^2 L_1 C_1 & -i\omega L_1 & 0 & 0 \\
-i\omega C_1 & 1 & 0 & 0 \\
0 & 0 & 1 - \omega^2 L_2 C_2 & -i\omega L_2 \\
0 & 0 & -i\omega C_2 & 1
\end{bmatrix}.
\]  
(D.6)

D.3 Coupled Dual Transmission Line Circuit Model

Fig. D.3 depicts the circuit model of the coupled transmission line pair within a printed DBE layout. To simplify the transfer matrix extraction, we make use of the symmetries in the circuit diagram. For that, we observe that the circuit seen between ports 1-4 and 2-4 are identical to each other. Likewise, the circuitry between ports 1-4 and 2-3 is equal. These symmetry relations imply that \( 4 \times 4 \) \( \mathbf{T}_C \) includes only two unique \( 2 \times 2 \) submatrices and has the form

\[
\begin{bmatrix}
V_1 \\
I_1 \\
V_2 \\
I_2
\end{bmatrix} = \begin{bmatrix}
T_{11} & T_{12} & T_{13} & T_{14} \\
T_{21} & T_{22} & T_{23} & T_{24} \\
T_{13} & T_{14} & T_{11} & T_{12} \\
T_{14} & T_{24} & T_{21} & T_{22}
\end{bmatrix}
\begin{bmatrix}
V_3 \\
I_3 \\
V_4 \\
I_4
\end{bmatrix}.
\]  
(D.7)
To extract the matrix elements of $\mathbf{T}_C$, we start by terminating the output ports with open circuit. This condition implies $I_3 = I_4 = 0$, and (D.7) becomes

\begin{align}
V_1 &= T_{11}V_3 + T_{13}V_4 \quad (D.8a) \\
I_1 &= T_{21}V_3 + T_{23}V_4 \quad (D.8b) \\
V_2 &= T_{13}V_3 + T_{11}V_4 \quad (D.8c) \\
I_2 &= T_{23}V_3 + T_{21}V_4. \quad (D.8d)
\end{align}

To solve the above equations, we first excite port 1 with $V_1 = 2$ and short circuit port 2 as $V_2 = 0$. For a simpler analysis, we further express this excitation as a superposition of two different modes. In the even mode, the excitation has the form $V_1 = V_1^e = 1$ and $V_2 = V_2^e = 1$. On the other hand, odd mode excitation is $V_1 = V_1^o = 1$ and $V_2 = V_2^o = -1$. Since no current flow occurs across the coupling capacitor for the even mode excitation, the circuit model can be simplified as shown in Fig. D.4 Applying Kirchoff’s voltage and current laws in the even mode circuit...
Figure D.4: Circuit model in Fig. D.3 when output ports are open and even mode is excited. \( L_M \) are doubled due to the coherent mutual inductance. \( C_M \) is removed since no current flows across it.

Figure D.5: Circuit model in Fig. D.3 when output ports are open and odd mode is excited. \( L_M \) vanishes due to negative mutual inductance. \( C_M \) is doubled due to virtual ground at the middle.

results in

\[
V_3^e = V_4^e = \frac{1}{1 - \omega^2(L_3 + 2L_M)C_3}, \quad (D.9a)
\]

\[
I_1^e = I_2^e = \frac{-i\omega C_3}{1 - \omega^2(L_3 + 2L_M)C_3}. \quad (D.9b)
\]

Likewise, Fig. D.5 depicts the simplified circuit diagram when the odd mode is
excited. In this case, Kirchoff’s voltage and current laws leads to

\[
V_3^o = -V_4^o = \frac{1}{1 - \omega^2(C_3 + 2C_M)L_3}, \quad \text{(D.10a)}
\]

\[
I_1^o = -I_2^o = \frac{-i\omega(C_3 + 2C_M)}{1 - \omega^2(C_3 + 2C_M)L_3}. \quad \text{(D.10b)}
\]

Next, we superimpose the even-odd mode results in (D.9) and (D.10) to determine the total voltages and currents as

\[
V_1 = V_1^e + V_1^o = 2, \quad \text{(D.11a)}
\]

\[
V_2 = V_2^e + V_2^o = 0, \quad \text{(D.11b)}
\]

\[
V_3 = V_3^e + V_3^o = \frac{1}{1 - \omega^2L_T C_3} + \frac{1}{1 - \omega^2C_T L_3}, \quad \text{(D.11c)}
\]

\[
V_4 = V_4^e + V_4^o = \frac{1}{1 - \omega^2L_T C_3} - \frac{1}{1 - \omega^2C_T L_3}, \quad \text{(D.11d)}
\]

\[
I_1 = I_1^e + I_1^o = \frac{-i\omega C_3}{1 - \omega^2L_T C_3} + \frac{-i\omega C_T}{1 - \omega^2C_T L_3}, \quad \text{(D.11e)}
\]

\[
I_2 = I_2^e + I_2^o = \frac{-i\omega C_3}{1 - \omega^2L_T C_3} - \frac{-i\omega C_T}{1 - \omega^2C_T L_3}, \quad \text{(D.11f)}
\]

where \( C_T = C_3 + 2C_M \) and \( L_T = L_3 + 2L_M \). After substituting the above values into (D.8) and solving the equation system, we obtain the unknown parameters of \( \mathbf{T}_C \) as

\[
T_{11} = 1 - \omega^2(L_3C_3 + L_3C_M + L_M C_3), \quad \text{(D.12a)}
\]

\[
T_{13} = \omega^2(L_3 C_M - L_M C_3), \quad \text{(D.12b)}
\]

\[
T_{21} = -i\omega(C_3 + C_M), \quad \text{(D.12c)}
\]

\[
T_{23} = i\omega C_M. \quad \text{(D.12d)}
\]

To extract the remaining terms of \( \mathbf{T}_C \), we next consider the case when the output ports of the coupled circuit in Fig. D.3 is shorted. This condition implies \( V_3 = V_4 = 0 \)
and (D.7) becomes

\[ V_1 = T_{12}I_3 + T_{14}I_4 \quad \text{(D.13a)} \]
\[ I_1 = T_{22}I_3 + T_{24}I_4 \quad \text{(D.13b)} \]
\[ V_2 = T_{14}I_3 + T_{12}I_4 \quad \text{(D.13c)} \]
\[ I_2 = T_{24}I_3 + T_{22}I_4. \quad \text{(D.13d)} \]

When the even mode excitation is applied to the input ports, the circuit can be
simplified as shown in Fig. D.6. From this circuit we immediately observe that

\[ I_e^1 = I_e^2 = I_e^3 = I_e^4 = \frac{1}{-i\omega(L_3 + 2L_M)}. \]  

(D.14)

Likewise, in the odd mode excitation we simplify the circuit as in Fig. D.7 and obtain

\[ I_o^1 = -I_o^2 = I_o^3 = -I_o^4 = \frac{1}{-i\omega L_3}. \]  

(D.15)

Superposition of the even-odd mode current and voltages in (D.14) and (D.15) leads to

\[ V_1 = V_e^1 + V_o^1 = 2, \]  

(D.16a)

\[ V_2 = V_e^2 + V_o^2 = 0, \]  

(D.16b)

\[ I_1 = I_e^1 + I_o^1 = \frac{1}{-i\omega L_T} + \frac{1}{-i\omega L_3}, \]  

(D.16c)

\[ I_2 = I_o^1 + I_o^2 = -\frac{1}{i\omega L_T} - \frac{1}{i\omega L_3}, \]  

(D.16d)

\[ I_3 = I_e^3 + I_o^3 = -\frac{1}{i\omega L_T} - \frac{1}{i\omega L_3}, \]  

(D.16e)

\[ I_4 = I_o^4 + I_o^4 = -\frac{1}{i\omega L_T} - \frac{1}{i\omega L_3}. \]  

(D.16f)

After substituting above variables into (D.13) and solving the equation system, the rest of the terms in \( T_C \) are obtained as

\[ T_{12} = -i\omega(L_3 + L_M), \]  

(D.17a)

\[ T_{14} = -i\omega L_M, \]  

(D.17b)

\[ T_{22} = 1, \]  

(D.17c)

\[ T_{24} = 0. \]  

(D.17d)
Using the matrix terms given in (D.23) and (D.17) in (D.7), full form of $\mathbf{T}_C$ can be written as

$$
\mathbf{T}_C = \begin{bmatrix}
1 - \omega^2 \Psi_1 & -i\omega \Psi_3 & \omega^2 \Psi_2 & -i\omega L_M \\
-i\omega \Psi_4 & 1 & i\omega C_3 & 0 \\
\omega^2 \Psi_2 & -i\omega L_M & 1 - \omega^2 \Psi_4 & -i\omega \Psi_3 \\
i\omega C_3 & 0 & -i\omega \Psi_4 & 1
\end{bmatrix},
$$  
(D.18)

where $\Psi_1 = (L_3 C_3 + L_3 C_M + L_M C_3)$, $\Psi_2 = (L_3 C_M - L_M C_3)$, $\Psi_3 = (L_3 + L_M)$, and $\Psi_4 = (C_3 + C_M)$.

### D.4 Partially Coupled Triple Line Circuit Model

Fig. D.8 represents the circuit model of partially coupled triple transmission lines. The transfer matrix $\mathbf{T}_{ML}$ of the circuit satisfies the relation

$$
\begin{bmatrix}
V_1 \\
I_1 \\
V_2 \\
I_2 \\
V_3 \\
I_3 \\
\vdots \\
T_{61} \\
\vdots \\
\vdots \\
T_{66}
\end{bmatrix} =
\begin{bmatrix}
\mathbf{T}_{11} & \mathbf{T}_{12} & \cdots & \cdots & \mathbf{T}_{16} \\
\mathbf{T}_{21} & \ddots & & & \vdots \\
\vdots & & \ddots & & \vdots \\
\vdots & & \vdots & \ddots & \vdots \\
\vdots & & \vdots & & \ddots \\
\mathbf{T}_{61} & \cdots & \cdots & \cdots & \mathbf{T}_{66}
\end{bmatrix}
\begin{bmatrix}
V_4 \\
I_4 \\
V_5 \\
I_5 \\
V_6 \\
I_6
\end{bmatrix}.
$$  
(D.19)

To extract $\mathbf{T}_{ML}$, we begin by forming the transfer matrices of the individual network sections marked in Fig. D.8. Noting that branches in the uncoupled line sections are not connected to each other, we can simply invoke transfer matrix (D.4) of a regular transmission line circuit (see Fig. D.1) to generate the transfer matrices of the first ($\mathbf{T}_{UC1}$) and second ($\mathbf{T}_{UC2}$) uncoupled sections as

$$
\mathbf{T}_{UCj} =
\begin{bmatrix}
1 - \omega^2 L_{j1} C_{j1} & -i\omega L_{j1} & 0 & 0 & 0 & 0 \\
-i\omega C_{j1} & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 - \omega^2 L_{j2} C_{j2} & -i\omega L_{j2} & 0 & 0 \\
0 & 0 & -i\omega C_{j2} & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 - \omega^2 L_{j3} C_{j3} & -i\omega L_{j3} \\
0 & 0 & 0 & 0 & -i\omega C_{j3} & 1
\end{bmatrix},
$$  
(D.20)
Figure D.8: Circuit model of the partially coupled triple transmission lines. The circuit is formed by cascading three uncoupled lines with three capacitively coupled lines. The coupled line section is composed of uncoupled and capacitive coupling blocks.

where \( j = 1 \) or \( j = 2 \). On the other hand, to extract transfer matrix \( T_{3C} \) of the coupling capacitor network, we first consider the circuit when the output ports are left open. Since this condition implies \( I_4 = I_5 = I_6 = 0 \), (D.19) becomes

\[
\begin{align*}
V_1 &= T_{11}V_4 + T_{13}V_5 + T_{15}V_6 \tag{D.21a} \\
I_1 &= T_{21}V_4 + T_{23}V_5 + T_{25}V_6 \tag{D.21b} \\
V_2 &= T_{31}V_4 + T_{33}V_5 + T_{35}V_6 \tag{D.21c} \\
I_2 &= T_{41}V_4 + T_{43}V_5 + T_{45}V_6 \tag{D.21d} \\
V_3 &= T_{51}V_4 + T_{53}V_5 + T_{55}V_6 \tag{D.21e} \\
I_3 &= T_{61}V_4 + T_{63}V_5 + T_{65}V_6. \tag{D.21f}
\end{align*}
\]

After the coupling capacitor network is redrawn as in Fig. D.9(a), applying Kirchhoff’s
current rule and Ohm’s law at nodes 4, 5, and 6 results in

\[
V_1 = V_4 \quad \text{(D.22a)}
\]

\[
I_1 = -i\omega(C_{M1} + C_{M3})V_4 + i\omega C_{M1}V_5 + i\omega C_{M3}V_6 \quad \text{(D.22b)}
\]

\[
V_2 = V_5 \quad \text{(D.22c)}
\]

\[
I_2 = i\omega C_{M1}V_4 - i\omega(C_{M1} + C_{M2})V_5 + i\omega C_{M2}V_6 \quad \text{(D.22d)}
\]

\[
V_3 = V_6 \quad \text{(D.22e)}
\]

\[
I_3 = i\omega C_{M3}V_4 + i\omega C_{M2}V_5 - i\omega(C_{M2} + C_{M3})V_6. \quad \text{(D.22f)}
\]

Comparing (D.22) with (D.21) shows that the unknown coefficients in (D.21) are
\[
T_{11} = 1 \quad T_{13} = 0 \quad T_{15} = 0 \\
T_{21} = -i\omega(C_{M1} + C_{M3}) \quad T_{23} = i\omega C_{M1} \quad T_{25} = i\omega C_{M3} \\
T_{31} = 0 \quad T_{33} = 1 \quad T_{35} = 0 \\
T_{41} = i\omega C_{M1} \quad T_{43} = -i\omega(C_{M1} + C_{M2}) \quad T_{45} = i\omega C_{M2} \\
T_{51} = 0 \quad T_{53} = 0 \quad T_{55} = 1 \\
T_{61} = i\omega C_{M3} \quad T_{63} = i\omega C_{M2} \quad T_{65} = -i\omega(C_{M2} + C_{M3}). \quad (D.23)
\]

Next, we consider the case when the output ports of the capacitive network is short circuited. This condition implies \(V_4 = V_5 = V_6 = 0\) and (D.19) becomes

\[
\begin{align*}
V_1 & = T_{12}I_4 + T_{14}I_5 + T_{16}I_6 \\
I_1 & = T_{22}I_4 + T_{24}I_5 + T_{26}I_6 \\
V_2 & = T_{32}I_4 + T_{34}I_5 + T_{36}I_6 \\
I_2 & = T_{42}I_4 + T_{44}I_5 + T_{46}I_6 \\
V_3 & = T_{52}I_4 + T_{54}I_5 + T_{56}I_6 \\
I_3 & = T_{62}I_4 + T_{64}I_5 + T_{66}I_6. \\
\end{align*}
\]

From the simplified circuit in Fig. D.9(b), we observe that

\[
\begin{align*}
V_1 & = V_4 = 0 \\
I_1 & = I_4 \\
V_2 & = V_5 = 0 \\
I_2 & = I_5 \\
V_3 & = V_6 = 0 \\
I_3 & = I_6. \\
\end{align*}
\]

134
Comparing (D.25) with (D.24) shows that the unknown coefficients in (D.24) are

\[
\begin{align*}
T_{12} &= 0 & T_{14} &= 0 & T_{16} &= 0 \\
T_{22} &= 1 & T_{24} &= 0 & T_{26} &= 0 \\
T_{32} &= 0 & T_{34} &= 0 & T_{36} &= 0 \\
T_{42} &= 0 & T_{44} &= 1 & T_{46} &= 0 \\
T_{52} &= 0 & T_{54} &= 0 & T_{56} &= 0 \\
T_{62} &= 0 & T_{64} &= 0 & T_{66} &= 1. \tag{D.26}
\end{align*}
\]

Finally, using the coefficients derived in (D.23) and (D.17), \( \overline{T}_{3C} \) can explicitly be written as

\[
\overline{T}_{3C} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
-i\omega(C_{M1} + C_{M3}) & 1 & i\omega C_{M1} & 0 & i\omega C_{M3} & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
i\omega C_{M1} & 0 & -i\omega(C_{M1} + C_{M2}) & 1 & i\omega C_{M2} & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
i\omega C_{M3} & 0 & i\omega C_{M2} & 0 & -i\omega(C_{M2} + C_{M3}) & 1
\end{bmatrix}. \tag{D.27}
\]

As usual, transfer matrix of the overall circuit can be obtained by multiplying individual transfer matrices as \( \overline{T}_{ML} = \overline{T}_{UC1}\overline{T}_{UC2}\overline{T}_{3C} \).


141


